AD621710

DREXEL INSTITUTE OF TECHNOLOGY

DIT Report No. 125-7

SOLUTION OF BLAST WAVES BY THE METHOD OF CHARACTERISTICS

by

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September 1965

This work was sponsored by the
Ballistic Research Laboratories
Aberdeen Proving Ground

under

Contract No. DA-36-034-ORD-3672 RD

TABLE OF CONTENTS

ABSTRACT

The numerical method of characteristics is applied to the plane, cylindrical, and spherical blast wave problems. The calculation begins at a constant time line within the blast wave. Along this line, all three flow variables are prescribed according to the similarity solution. Along a fixed back boundary, which lies between the shock front and the time-axis, one flow variable is prescribed. On the other boundary, the strong shock equations apply.

When compared to the exact similarity solution, the results of the method of characteristics are found to be accurate to within 1%, for all variables along the shock front, after a pressure drop of 99%. Also, h^2-type extrapolation of calculated results always improves the accuracy, whereas the h-type extrapolation may not.

NOMENCLATURE

c = sound speed

E = parameter proportional to the total energy within the wave

h = parameter proportional to the mesh size

k = x_b/x_s position ratio of back boundary

P = pressure

t = time variable

u = particle velocity

U = shock wave velocity

x = space variable

γ = specific heat ratio

ν = geometric parameter (ν = 1, 2, or 3 corresponds to plane, cylindrical, or spherical flow, respectively)

ρ = density

ρ_1 = constant density outside wave zone

Subscripts

s = at the shock front

b = at the back boundary

I. INTRODUCTION

In the hydrodynamic regime of hypervelocity impact problems, the governing equations are the conservation equations for unsteady, compressible fluid flow. Due to the nonlinearity of these equations, very few closed form analytical solutions exist and, for most problems, numerical methods are the only resort. Among the numerical methods, the two most commonly used are the finite-difference method and the method of characteristics. In the finite-difference method, an artificial viscosity must be added, either explicitely, as in the von Neuman-Richtmyer q method[1], or implicitely, as in the Lax method[2]. By the introduction of artificial viscosity, discontinuities (shocks) are spread into narrow regions, across which flow variables change rapidly but continuously. Flow fields with a complicated shock geometry can be calculated without extra difficulty. Because of the smearing of shocks, however, the finite-difference method is not accurate, especially in regions near the shock front. Also, it does not give an exact treatment of singularities such as centered rarefaction waves. As mentioned in Reference 3, two of the finite-difference calculations of hypervelocity impact problems gave initial peak pressures which differed from the exact values by at least 15%. In the method of characteristics, the shock is calculated by the exact shock equations; therefore, better accuracy can be expected. For problems with simple geometry, the method of characteristics seems to be more suitable.

For solving one-dimensional unsteady flow problems, the method of characteristics as described in textbooks such as Courant and Friedrichs[4], Shapiro[5], and Rudinger[6] has been used widely. Most recent accounts of

this method, with special emphasis on high speed computation by electronic computers may be found in books by Fox[7] and Hoskin[8]. Due to the nonlinearity of the equations, an exact analysis of the error involved in the numerical calculation by either the method of characteristics or the finite-difference method is very difficult. Makino and Shear[9] showed that replacing the differential characteristic equations by difference equations introduces an error of the h^2-type. But the type of error in the variables solved from these difference characteristic equations is still not known. Without an exact error analysis, the accuracy of any numerical method can be estimated by comparing the result either with accurate experimental results or with an exact analytical solution.

In the present paper, calculations of the blast wave problems are made using the numerical method of characteristics, and the results are compared with the exact similarity solutions. If the region in which the density is close to zero is excluded, the method of character-isitcs yields results which are accurate to within 1% on the shock front when a 99% pressure decrease is reached. Different types of extrapolation formulas are applied to three sets of calculations of the same problem, each set having a different mesh size. It is found that the h^2-type extrapolation always improves the results.

These results of the blast wave calculations indicate that for other unsteady flow problems which do not include extremely low density regions, the numerical method of characteristics can be expected to be of the same accuracy.

For the continuity of presentation, the basic governing equations and the corresponding characteristic equations are given in section II; different similarity solutions for blast waves are discussed in

section III. Discussions on the formulation of the problem, numerical calculations, and method of extrapolation are then presented in the second half of the paper. Appendices A, B, and C are included mainly for easy references although some of the material represents new ways of presenting these topics. Appendix D includes an 8-digit table of properties within the blast waves. This table may be used to evaluate the accuracy of other numerical methods.

II. GOVERNING EQUATIONS

When the unsteady flow of an ideal gas depends on only one space coordinate, the governing equations in Eulerian coordinates for continuous flow without friction and heat transfer are the continuity equation

$$\frac{\partial\rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial\rho}{\partial x} + (\nu-1)\frac{\rho u}{x} = 0 \qquad (1)$$

the momentum equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \qquad (2)$$

and the energy equation

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^{\gamma}}\right) + u\frac{\partial}{\partial x}\left(\frac{p}{\rho^{\gamma}}\right) = 0 \qquad (3)$$

where ν has a value of 1, 2, or 3 corresponding to plane, cylindrical, or spherical flows, respectively. From this system of hyperbolic equations, the following three physical characteristic equations may be obtained (see Appendix A).

Along I and II $\qquad \frac{dx}{dt} = u \pm c \qquad (4)$

Along III $\qquad \frac{dx}{dt} = u \qquad (5)$

where I and II are the right-traveling and left traveling waves, respectively, and III is the particle path line. The corresponding state characteristics are

along I and II, $$du = \mp \frac{1}{\rho c} (dp) \mp (\nu-1) \frac{uc}{x} (dt) \tag{6}$$

along III, $$d\rho = \frac{1}{c^2}(dp) \tag{7}$$

In these characteristic equations, the following expression for sound speed as a function of pressure and density has been introduced:

$$c^2 = \gamma \frac{p}{\rho} \tag{8}$$

After eliminating c^2 by using eq. (8), eq. (7) may be integrated, and the result indicates that p/ρ^γ , or the entropy, remains constant along each particle path. However, this constant is in general different for each particle path. Notice that this assumption is incorporated in the energy equation (3). It is convenient to eliminate the density ρ from the characteristic equations by using equation (8). The state characteristics are then

along I and II, $$du = \mp \frac{c}{\gamma p} (dp) \mp (\nu-1) \frac{uc}{x} (dt) \tag{9}$$

along III, $$\frac{p}{p_o} = \left(\frac{c}{c_o}\right)^{\frac{2\gamma}{\gamma-1}} \tag{10}$$

The constants c_o and p_o must be evaluated for each particle path. Equations (4), (5), (9), and (10) are the basic equations which are to be used in the method of characteristics.

Only shock waves of very strong intensity will be used in our calculation. This restriction is consistant with the similarity solution of blast waves, the results of which will be compared with our method of characteristics solution. The Rankine-Hugoniot relations for a strong shock moving into a uniform, stagnant region are

$$u_s = \frac{2}{\gamma+1} U \tag{11}$$

$$c_s = \sqrt{\frac{2\gamma(\gamma-1)}{(\gamma+1)}} U \tag{12}$$

$$p_s = \frac{2}{\gamma+1} \rho_1 U^2 \tag{13}$$

In the numerical procedure of the method of characteristics, all governing equations are put into finite-difference form. Thus, equations (4), (5), (9), and (10) become

along I and II,

$$\frac{\Delta x}{\Delta t} = \bar{u} \pm \bar{c} \tag{14}$$

$$\Delta u = \mp (\frac{\bar{c}}{\bar{p}}) \frac{\Delta p}{\gamma} \mp (\nu-1)(\frac{\bar{u}\bar{c}}{\bar{x}})(\Delta t) \tag{15}$$

along III,

$$\frac{\Delta x}{\Delta t} = \bar{u} \tag{16}$$

$$\frac{p}{p_o} = (\frac{c}{c_o})^{\frac{2\gamma}{\gamma-1}} \tag{17}$$

where Δ represents the (central) difference between two adjacent points along a characteristic and the barred values represent the average between the same two points.

III. SIMILARITY SOLUTIONS OF BLAST WAVES

The self-similar solutions of blast waves have been analyzed by many investigators and are among the few known exact solutions of the one-dimensional, non-isentropic (entropy changes from particle to particle), unsteady flows. This type of flow will be calculated by the numerical method of characteristics and the results compared with the analytical solutions.

Taylor[10] solved the blast wave problem which consists of the instantaneous release of a finite amount of energy at a point within a stagnant, ideal gas field. The subsequent flow was non-isentropic, with spherical symmetry ($u = 0$ at $x = 0$), and with a strong shock propagating into the stagnant gas. By utilizing the self-similar property of the flow, the governing partial differential equations were reduced to ordinary differential equations with one independent variable. These equations were then integrated numerically and the results presented in tabular form. Lin[11] followed essentially the same technique and solved the problem of the instantaneous release of energy along a line and the ensuing cylindrically symmetric flow. His results in tabular form were also obtained by numerical integration. Recently, Lee[12] presented a unified approach which encompasses a large number of similarity solutions. He did not attempt to integrate the resulting ordinary differential equations.

Sedov[13] and Rouse[14], among others, have successfully integrated the ordinary differential equations of the blast wave problems and obtained closed form solutions. They also included in their solutions the case of plane motion, in addition to the cylindrical and spherical cases. Since these solutions are in closed form, their numerical value can be calculated to any desired degree of accuracy without too much difficulty. On the other hand, the solutions of Taylor and Lin involve numerical integration; therefore, a high degree of accuracy is difficult to achieve. Figure 1 gives a comparison of the results of our calculation of Sedov's spherical wave solution and the solution obtained by Taylor. It can be seen that, although Taylor's values may be satisfactory for other purposes, they are not accurate enough to serve

as a standard to evaluate the accuracy of other numerical methods. Figure 10 of Reference 14 gives a similar comparison of the analytical solution of Rouse for the cylindrical case and the numerically integrated solution of Lin.

It is evident, then, that only accurately evaluated closed form solutions can be used as a standard to gauge the accuracy of numerical methods. Gerber and Bartos[15] presented a numerical table calculated from Rouse's equations for cylindrical waves with $\gamma = 7/5$ and $5/3$. However, their table contains too few data points for our purposes. For easy reference, we have made numerical calculations of Sedov's solution for the plane, cylindrical, and spherical cases with two values of γ, 7/5 and 5/3. These numerical results are tabulated in Appendix D, table II. Eight significant figures were used in all calculations. A comparison with calculations made for a few points using twenty-two digits indicated that the data in Appendix D are accurate at least to the sixth digit. Sedov's solution, for $\gamma = 7/5$, is as follows,

$$\frac{x}{x_s} = \left[\frac{3(\nu+2)V}{5}\right]^{-\frac{2}{2+\nu}} \left[\frac{21(\nu+2)V-30}{5}\right]^{\alpha_2} \left[\frac{3(\nu+2)}{2\nu+1}\left(1-\frac{(5+\nu)V}{5}\right)\right]^{-\alpha_1} \tag{18}$$

$$\frac{u}{u_s} = \frac{3(\nu+2)V}{5}\frac{x}{x_s} \tag{19}$$

$$\frac{\rho}{\rho_s} = \left[\frac{21(\nu+2)V-30}{5}\right]^{\alpha_3} \left[6-3(\nu+2)V\right]^{-\frac{10}{3}} \left[\frac{3(\nu+2)}{2\nu+1}\left(1-\frac{(5+\nu)V}{5}\right)\right]^{\alpha_4} \tag{20}$$

$$\frac{p}{p_s} = \left[\frac{3(\nu+2)V}{5}\right]^{\frac{2\nu}{2+\nu}} \left[6-3(\nu+2)V\right]^{-\frac{7}{3}} \left[\frac{3(\nu+2)}{2\nu+1}\left(1-\frac{(5+\nu)V}{5}\right)\right]^{\alpha_4-2\alpha_1} \tag{21}$$

where $\alpha_1 = \frac{2}{5+\nu}\left(\frac{11\nu^2+20\nu+14}{5\nu^2+14\nu+8}\right)$

$\alpha_2 = \frac{2}{4+5\nu}$

$$\alpha_3 = \frac{5\nu}{4+5\nu}$$

$$\alpha_4 = \frac{5\alpha_1(\nu+2)}{3}$$

and V is a parameter with the range of variation

$$\frac{10}{(\nu+2)7} \leq V \leq \frac{10}{(\nu+2)6}$$

The properties at the shock front, which are indicated by the subscript s in the above equations, are given by

$$u_s = \frac{5}{3(\nu+2)} \left(\frac{E}{\rho_1}\right)^{\frac{1}{2}} x_2^{-\frac{\nu}{2}} \tag{22}$$

$$\rho_s = 6\rho_1 \tag{23}$$

$$p_s = \frac{10E}{3(\nu+2)^2} x_2^{-\nu} \tag{24}$$

The position of the shock front, at any time t, is

$$x_s = \left(\frac{E}{\rho_1}\right)^{\frac{1}{2+\nu}} t^{\frac{2}{2+\nu}} \tag{25}$$

IV. FORMULATION OF THE PROBLEM FOR NUMERICAL CALCULATION

In applying the numerical method of characteristics to the blast wave problems, it is desirable to avoid the origin (x = 0, t = 0) and the time axis (x = 0). The origin is a singular point with unbounded values of pressure, particle velocity, and sound speed and, along the time axis, the sound speed is infinite. Only a region of the flow field which does not include the time axis will be calculated. This region is bounded by a constant (non-zero) time line on the bottom,

the shock front on the right, and a back boundary curve on the left with the equation $x_b = k\, x_s$, where k is a constant. (See Figure 2.) To pose a problem which is mathematically correct, the following boundary and initial conditions must be prescribed on the back boundary, shock front, and constant time line.

The constant time line, $t = t_o$ in Fig. 2, is considered as the initial value curve; therefore, values of all three dependent variables p, u, and c must be specified on it. These values were calculated from the exact solution, equations (18) to (25) and equation (8), and considered as the specified values for the solution by the method of characteristics. The back boundary curve is a "time-like" arc, according to the terminology used by Courant and Friedrichs[4], since two of the three characteristics, II and III, reach this curve from inside the region R. The correct boundary condition requires that one of the three dependent variables be specified on this curve. In our case, the particle velocity u was calculated from the exact solution and used as the one specified dependent variable.

The shock front is also a time-like arc, but in this case only the I-characteristic reaches a point on it from inside region R. Also, there is no prior information about its exact position. There are four unknown quantities to be determined at the shock front, U, p, c, and u. Therefore, the strong shock conditions, equations (11), (12), and (13), are the correct boundary conditions at the shock front since these three equations, together with the equation of the I-characteristic serve to determine the four unknown quantities. The position of the shock path is determined by integrating the shock velocity U.

V. NUMERICAL CALCULATION

In the numerical calculation, a number of equally spaced points on the constant time line, $t = t_0$, are chosen, and the properties at these points are calculated from the exact solution. A network is then constructed in the physical plane consisting of the I and II characteristics passing through these points and the reflections of these characteristics from the back boundary and shock front. Properties at the mesh points of this network will be determined. Since the equations for the state characteristics involve the time variable t, and the physical characteristics involve the state variables u and c, neither the physical nor the state plane can be independently constructed. An iteration scheme must be adopted to determine a mesh point in both the physical and state planes, i.e., for each point the five unknowns x, t, p, u, and c must be solved for simultaneously by some iteration process. The iteration scheme adopted here is described in detail in Appendix B. It follows essentially the standard procedure given in textbooks such as References 5 and 16. In brief, it involves the construction of the I and II characteristics from two neighboring points, locating their intersection, and tracing back a path line until it intersects the straight line connecting the original two points. At this intersection point the entropy is calculated by a linear interpolation of the values at the two known points.

The calculated results for plane, cylindrical, and spherical waves are presented in Table I. The same values of ρ_1 and E were used for all three cases. For the factor k, which determines the position of the back boundary, three different values, 0.5, 0.7, and 0.8 were used

TABLE I. INITIAL AND FINAL DATA OF BLAST WAVE PROBLEMS CALCULATED BY THE METHOD OF CHARACTERISTICS

ν	k	no. of initial data points	PROPERTIES ON SHOCK FRONT						
				Pressure drop %	x_s ft.	t_s sec.	p_s lb/ft^2	u_s ft/sec	c_s ft/sec
1 (plane wave)	0.5	101	initial		2.0	1.125×10^{-4}	2.867×10^{5}	9.876×10^{3}	5.226×10^{3}
			final	99.1	229.1	.1383	2.499×10^{3}	9.220×10^{2}	4.879×10^{2}
2 (cylindrical wave)	0.7	121	initial		2.0	1.591×10^{-4}	8.064×10^{4}	5.237×10^{3}	2.771×10^{3}
			final	98.4	15.6	9.690×10^{-3}	1.322×10^{3}	6.705×10^{2}	3.548×10^{2}
3 (spherical wave)	0.8	81	initial		2.0	2.250×10^{-4}	2.580×10^{4}	2.963×10^{3}	1.568×10^{3}
			final	99.0	9.4	1.088×10^{-3}	2.452×10^{2}	2.888×10^{2}	1.528×10^{2}

For all wave geometries ρ_1 = .0024497 slugs/ft^3, E = 1.5482649 x 10^6 ft-lb/ft$^{(3-\nu)}$.

for the plane, cylindrical, and spherical cases, respectively. As the value of k decreases, i.e., as a point moves towards the time axis, the value of the density ρ approaches zero very rapidly and the sound speed c increases rapidly without bound, as shown in Fig. 3. Because of these extreme values, the accuracy of the numerical solution becomes inadequate as k becomes smaller. The mesh of the characteristic network in the physical plane is of irregular size when these extreme values of c are included. In Figure 4 a coarse network for the case of $\nu = 2$, $k = 0.6$ is shown. For practical problems such as those encountered in hypervelocity impacts, exploding wires, or atmoshpheric explosions, the initial spacial distribution of c at any two points seldom differs by a ratio of more than 2.5, and the initial density is seldom close to zero. Therefore, the value of k was so chosen that the ratio of the sound speed at the back boundary over that at the shock front was approximately 2.5 for all three cases. This results in the aforementioned values of 0.5, 0.7, and 0.8 for the plane, cylindrical, and spherical cases, respectively, as indicated in Fig. 3. The initial position of the shock front was aribtrarily taken as 2 feet for all three cases; therefore a different value of t_o was obtained for each case.

The numerical procedure was programmed in Fortran II language and the calculation performed on an IBM 7040 computer. The maximum computation time was approximately two hours, which gave a pressure at the last point on the shock front of about 1% of the initial pressure on the shock.

The accuracy of these calculations is shown in Figures 5, 6, and 7. Figure 5(a) shows the physical plane of the plane wave, indicating

the position of the shock front, the back boundary, and the last line of numerical calculation. The percent relative error of p, u, and c at points on the last line of calculation as compared to the exact solution is plotted in Fig. 5(b). The percent error of t, p, u, and c along the shock front is shown in Fig. 5(c). These errors along the shock front were obtained by comparing the calculated values of these variables with those of the exact solution at the same x location. Since the location of the numerically calculated shock front deviates slightly from the exact shock front, the properties are not compared at the same value of time; however, this error in time is quite small. The maximum error along the shock front for all variables is less than .6%; while on the last line of calculation the maximum error is less that 3%. Similar results for the cylindrical and spherical cases are shown in Figures 6 and 7, respectively.

VI. EXTRAPOLATION OF NUMERICAL RESULTS

It is shown in Appendix C that the error introduced by replacing the differential equations (4), (5), and (9) by finite-difference equations (14), (15), and (16) is of the h^2-type, where h is a parameter proportional to the mesh size. Also, the linear interpolation process, which is used in the solution of the difference equations, is known to be correct to the order of h^2. The boundary conditions and initial conditions used in our problem are either prescribed functions or exact strong shock conditions. Since all errors introduced by the numerical solution are of the order h^2, it seems reasonable to assume that each of the numerically calculated variables has an error which is of the h^2-type, i.e.,

$$p_e - p = \phi_1 h^2 + \phi_2 h^4 + \phi_3 h^6 + \ldots \qquad (26)$$

where p_e is the exact value of pressure, p is the approximate pressure at the same point calculated with a mesh size h, and ϕ_i are quantities which are independent of the mesh size and depend only upon the location of the point in question. Equations similar to equation (26) are also assumed to hold for c and u.

With this order of error assumed, it is possible to apply Richardson's extrapolation method[17] to the calculated values at any point with two different mesh sizes and obtain new values which are more accurate. If two values of the pressure at a point are calculated, p_1 corresponding to a mesh size h_1, and p_2 corresponding to a mesh size h_2, then the extrapolated value may be determined from the following equations which are obtained by truncating equation (26),

$$\begin{aligned} p_e &= p_1 + \phi_1 h_1^2 \\ p_e &= p_2 + \phi_1 h_2^2 \end{aligned} \qquad (27)$$

Elimination of ϕ_1 from these equations gives

$$p_e = \frac{h_2^2 p_1 - h_1^2 p_2}{h_2^2 - h_1^2} \qquad (28)$$

In this equation, p_e represents the extrapolated value of p. Similarly, if calculations with three different mesh sizes, h_1, h_2, and h_3, are performed, the following equations may be used to solve for p_e,

$$\begin{aligned} p_e &= p_1 + \phi_1 h_1^2 + \phi_2 h_1^4 \\ p_e &= p_2 + \phi_1 h_2^2 + \phi_2 h_2^4 \\ p_e &= p_3 + \phi_1 h_3^2 + \phi_2 h_3^4 \end{aligned} \qquad (29)$$

It must be mentioned that, although the finite-difference equations involve errors of the h^2-type, there is no proof to show that the errors in the calculated values of p, u, and c themself are also of the h^2-type. Equation (26) is merely an assumption. However, for the present problem, as demonstrated in the numerical examples below, the h^2-type extrapolation does improve the numerical results.

Roberts [18] used an extrapolation formula which is based on the assumption that all errors in numerical solutions are of the h-type; or, in our notation, his assumption becomes

$$p_e - p = \psi_1 h + \psi_2 h^2 + \psi_3 h^3 + \ldots \qquad (30)$$

where ψ_i are functions independent of h and dependent only upon the position of the point in question. The resulting extrapolation formula for two sets of calculations is

$$p_e = \frac{h_2 p_1 - h_1 p_2}{h_2 - h_1} \qquad (31)$$

For comparison purposes, both h^2-type and h-type extrapolations are applied to the present numerical results for the plane wave case. In addition to the 101 initial point (100 initial segments) calculation, two more sets of calculations, using 50 and 25 initial segments, respectively, were performed. The mesh sizes of these three cases are designated as $h = 1/4$, $h = 1/2$, and $h = 1$, respectively. The errors in p along the shock front for these three cases are shown in Fig. 8. It can be seen that the error decreases as the mesh size decreases. Several sets of extrapolated values of the pressure along the shock front were obtained by using different combinations of the three sets of calculations with different mesh sizes. These extrapolated values of p were compared with the exact values and the error plotted in

Figure 8. The curve labeled $h^2(1, 1/2)$ represents the error in the extrapolated value of p obtained from the h^2-type formula, eq. (28), by using the pressures from the corresponding calculations with mesh sizes $h = 1$ and $h = 1/2$; $h(1, 1/2, 1/4)$ represents h-type extrapolation of pressures corresponding to mesh sizes $h = 1$, 1/2, and 1/4, etc. The $h(1/2, 1/4)$ extrapolation gives the least error, while the other three h-extrapolations give errors equal to or larger than the original calculation of mesh size $h = 1/4$. On the other hand, all four h^2-extrapolations give errors which are smaller than the $h = 1/4$ case. In other words, for the present problem, the h^2-extrapolations always improve the results, whereas the h-extrapolations do not necessarily improve the results. The results also indicate that extrapolations of three sets of calculations are not necessarily better than extrapolations of two sets of calculations. Extrapolation curves for the variables u and c give essentially the same results as those of pressure and, therefore, are not presented.

VII. CONCLUSION

The numerical method of characteristics produces very accurate results when applied to blast wave problems if the region where the density is close to zero is excluded. For other one-dimensional unsteady fluid flow problems which do not include zero density regions, it is reasonable to expect that the numerical method of characteristics will also produce accurate results.

According to our numerical calculations, h^2-type extrapolation always improves the accuracy, whereas h-type extrapolation may not.

For the purpose of evaluating the accuracy of any numerical method used for solving unsteady, non-isentropic flows, the similarity solution of blast waves may be used as a standard for comparison. Since the similarity solutions are quite lengthy, they have been computed for two values of γ, and the numerical results are presented in Appendix D for easy reference.

VIII. REFERENCES

[1]von Neumann, J. and Richtmyer, R. D., "A Method for the Numerical Calculations of Hydrodynamic Shocks," Jour. Appl. Phys., 21, 232-237 (1950).

[2]Lax, P. D., "Weak Solutions of Non-linear Hyperbolic Equations and Their Numerical Computations," Commun. Pure Appl. Math. 7, 159-193 (1954).

[3]Chou, P. C., Sidhu, H. S., and Zajac, L. J., "The Strong Plane Shock Produced by Hypervelocity Impact and Late-Stage Equivalence," Proc. of the Seventh Hypervelocity Impact Symposium, (Martin Company, Orlando, Florida, Feb. 1965) Vol. II., pp. 221-308.

[4]Courant, R. and Friedrichs, K. O., *Supersonic Flow and Shock Waves* (Interscience Publishers, New York, 1948), Chap. III, p. 86.

[5]Shapiro, A. H., *The Dynamics and Thermodynamics of Compressible Fluid Flow*, (Ronald Press, New York, 1954), Vol. II, Chap. 24, pp. 972-978.

[6]Rudinger, G., *Wave Diagrams for Unsteady Flow in Ducts*, (D. Van Nostrand Co., New York, 1955).

[7]Fox, L., *Numerical Solution of Ordinary and Partial Differential Equations*, (Pergamon Press, Oxford, 1962) Chap. 26, pp. 325-338.

[8]Hoskin, N. E., "Solution by Characteristics of the Equations of One-Dimensional Unsteady Flow," *Methods in Computational Physics, Advances in Research and Application*, edited by Alder, B., Fernbach, S., and Rotenberg, M., (Academic Press, New York, 1964), pp. 265-293.

[9]Makino, R. C. and Shear, R. E., "Unsteady Spherical Flow Behind A Known Shock Line," Ballistic Research Laboratories Report 1154 (November 1961).

[10]Taylor, G. I., "The Formation of a Blast Wave by a Very Intense Explosion, I Theoretical Discussion," Proc. Roy. Soc. (London) A201, 159-174 (1950).

[11]Lin, S. C., "Cylindrical Shock Waves Produced by Instantaneous Energy Release," J. Appl. Phys. 25, 54-57 (1954).

[12]Lee, T. S., "Unsteady Flows in an Unsteady Environment," Phys. Fluids 8, 1266-1272 (1965).

[13]Sedov, S. I., *Similarity and Dimensional Methods in Mechanics*, (Academic Press, New York, 1959), Chap. IV, p. 146.

[14]Rouse, C. A., "Theoretical Analysis of the Hydrodynamic Flow in Exploding Wire Phenomena," *Exploding Wires*, edited by Chace, W. G. and Howard, K. M. (Plenum Press, New York, 1959) pp. 227-263.

[15]Gerber, N. and Bartos, J. M., "Tables of Cylindrical Blast Functions for $\gamma = 5/3$ and $\gamma = 7/5$," Ballistic Research Laboratories Report 1376 (October 1961).

[16]Hartree, D. R., *Numerical Analysis* (Oxford University Press, London, 1958), 2nd Ed., chap. X, pp. 257-263.

[17]Richardson, L. F., "The Approximate Arithemetical Solutions by Finite Differences of Physical Problems Involving Differential Equations," Phil. Trans. Roy. Soc. (London) 210, 307 (1911).

[18]Roberts, L., "On the Numerical Solution of the Equations for Spherical Waves of Finite Amplitude, II," J. Math. Phys. 36, 329-337 (1959).

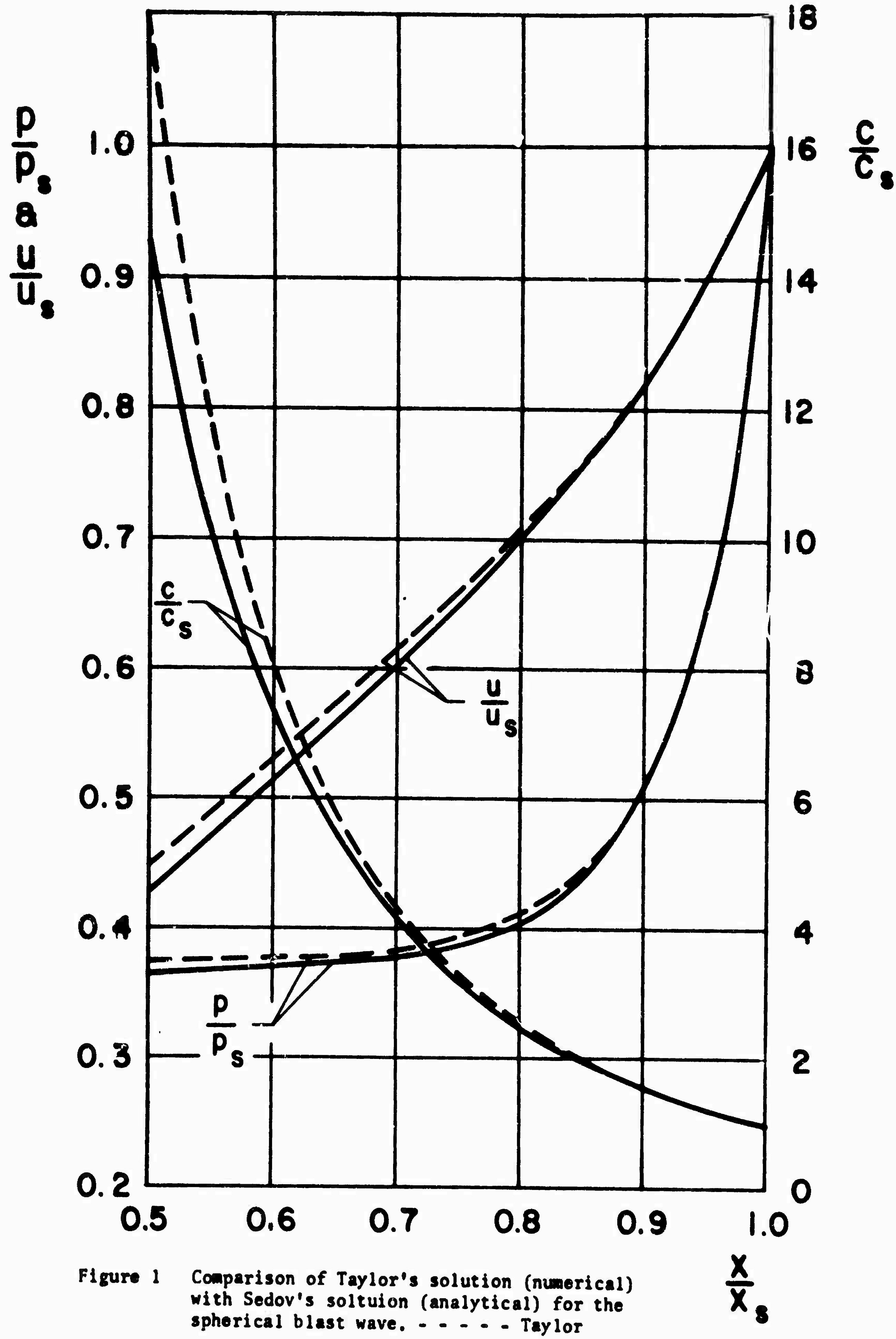

Figure 1 Comparison of Taylor's solution (numerical) with Sedov's soltuion (analytical) for the spherical blast wave. - - - - - Taylor ——— Sedov

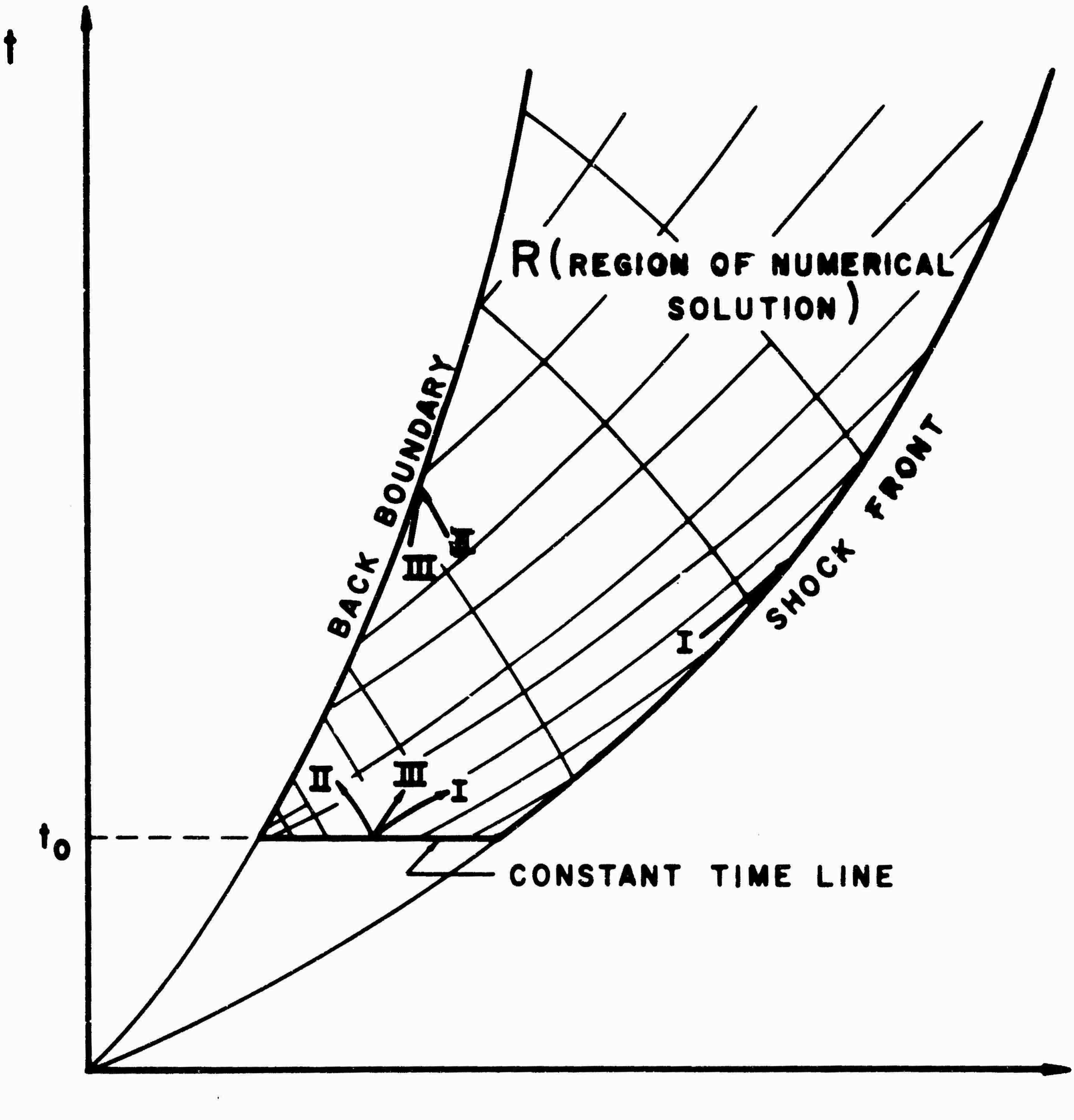

Figure 2 Physical Plane Showing Region of Numerical Solution for Blast Wave Problems; on the back boundary ($x_b = kx_s$), u is prescribed; on the constant time line, p, u, and c are prescribed; on the shock front, the strong shock equations (11) to (13) apply.

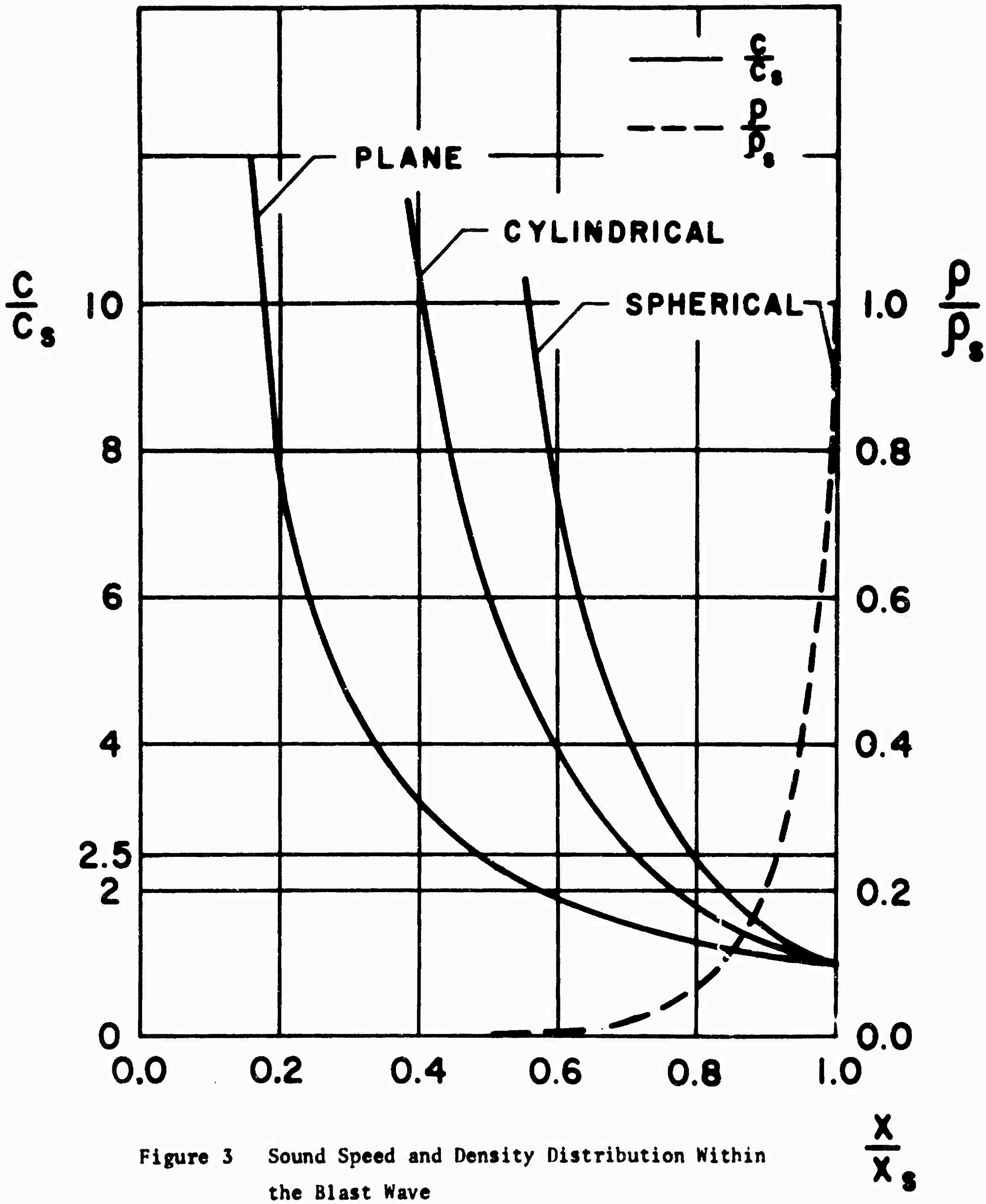

Figure 3 Sound Speed and Density Distribution Within the Blast Wave

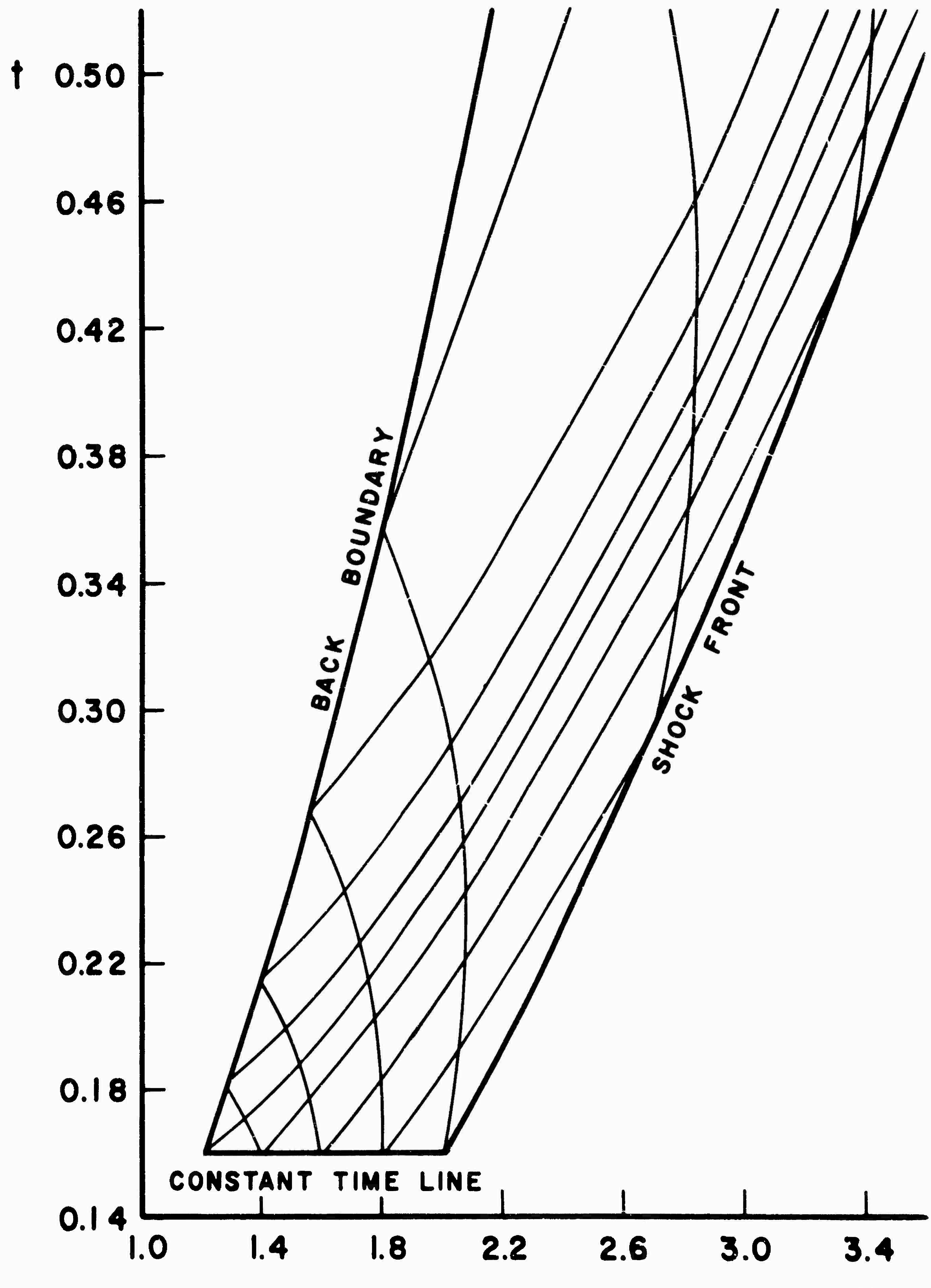

Figure 4 Physical Plane for Cylindrical Blast Wave Showing a Coarse Characteristic Net (k = .6)

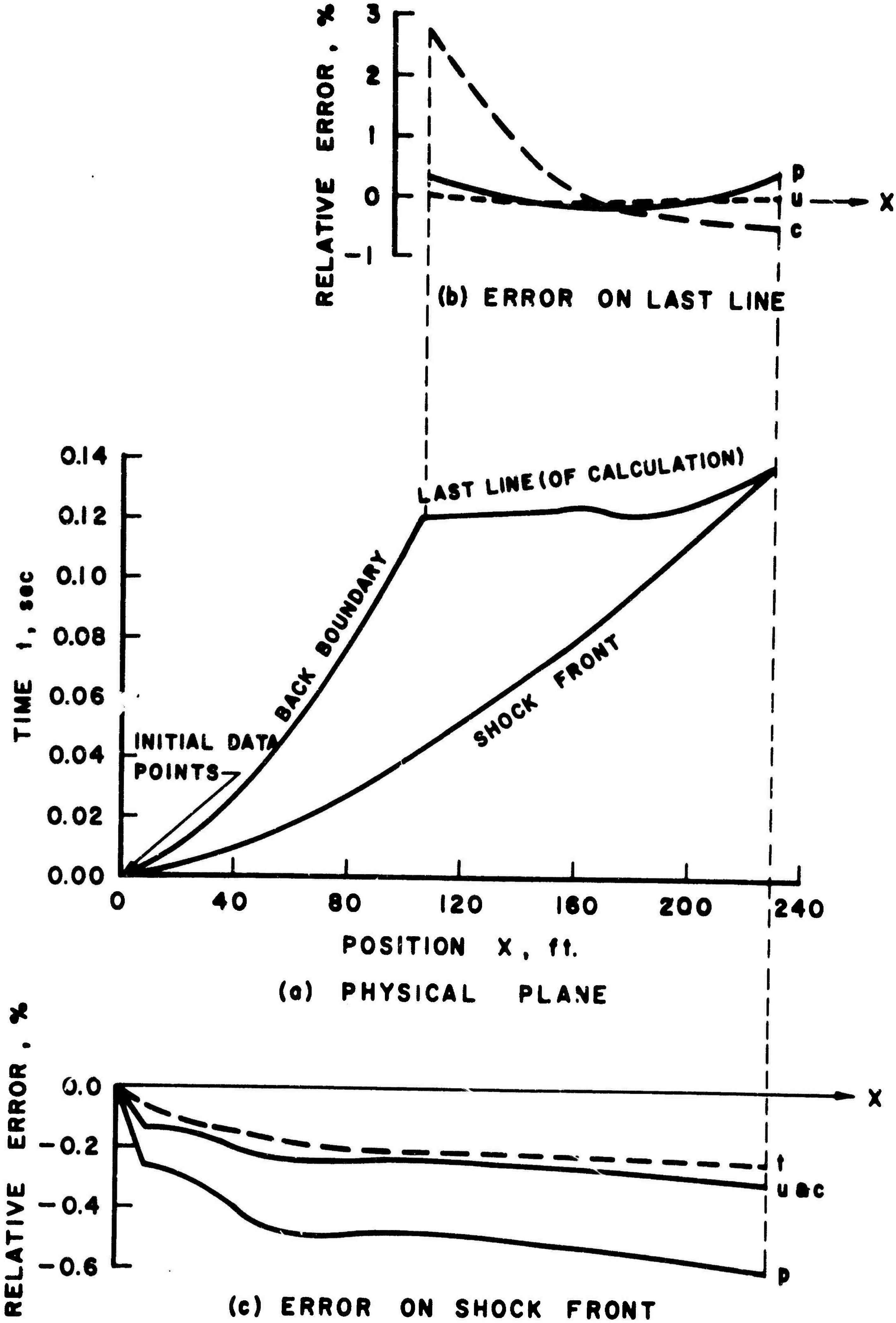

Figure 5 — Plane Wave Error Analysis

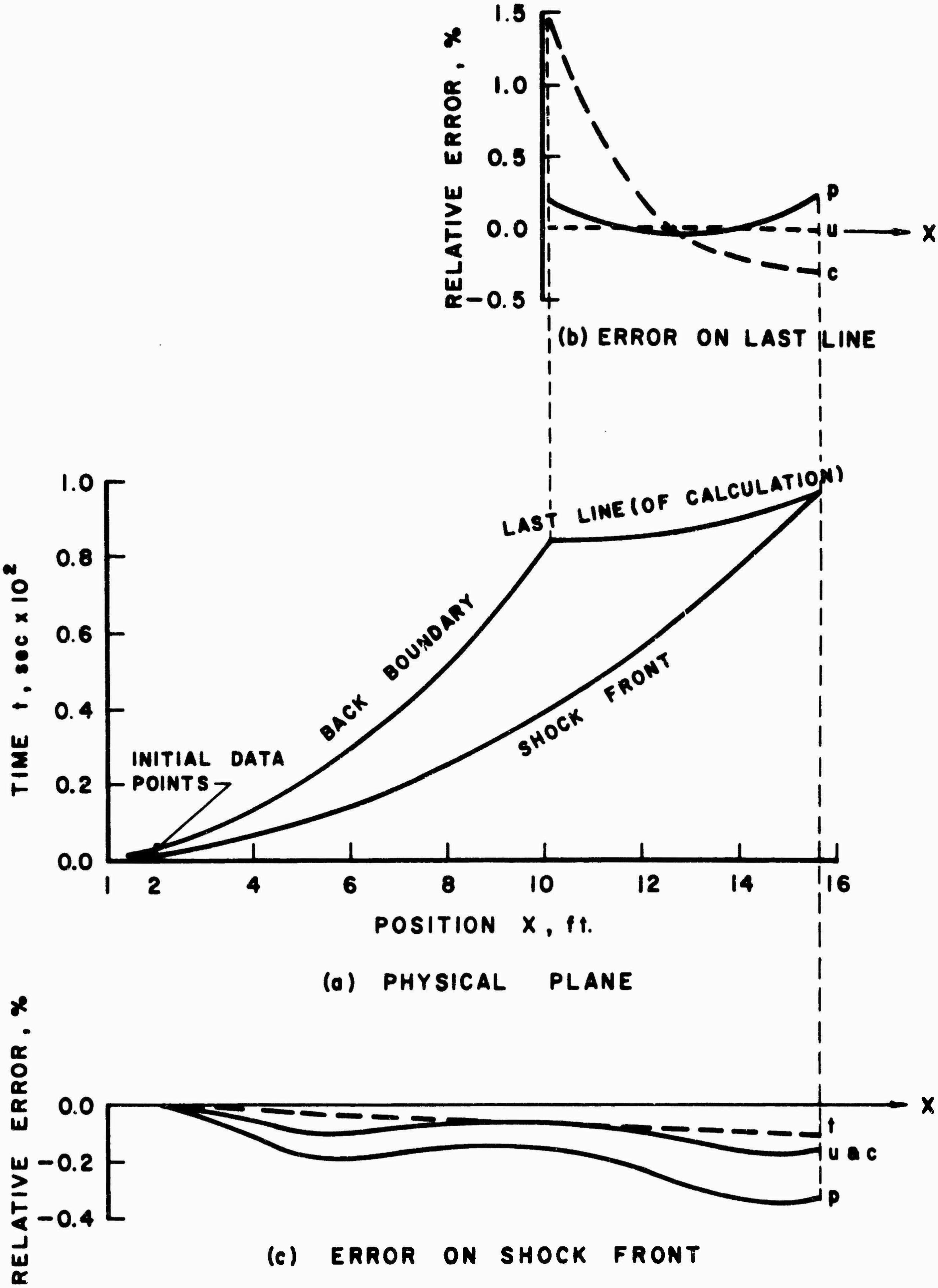

Figure 6— Cylindrical Wave Error Analysis

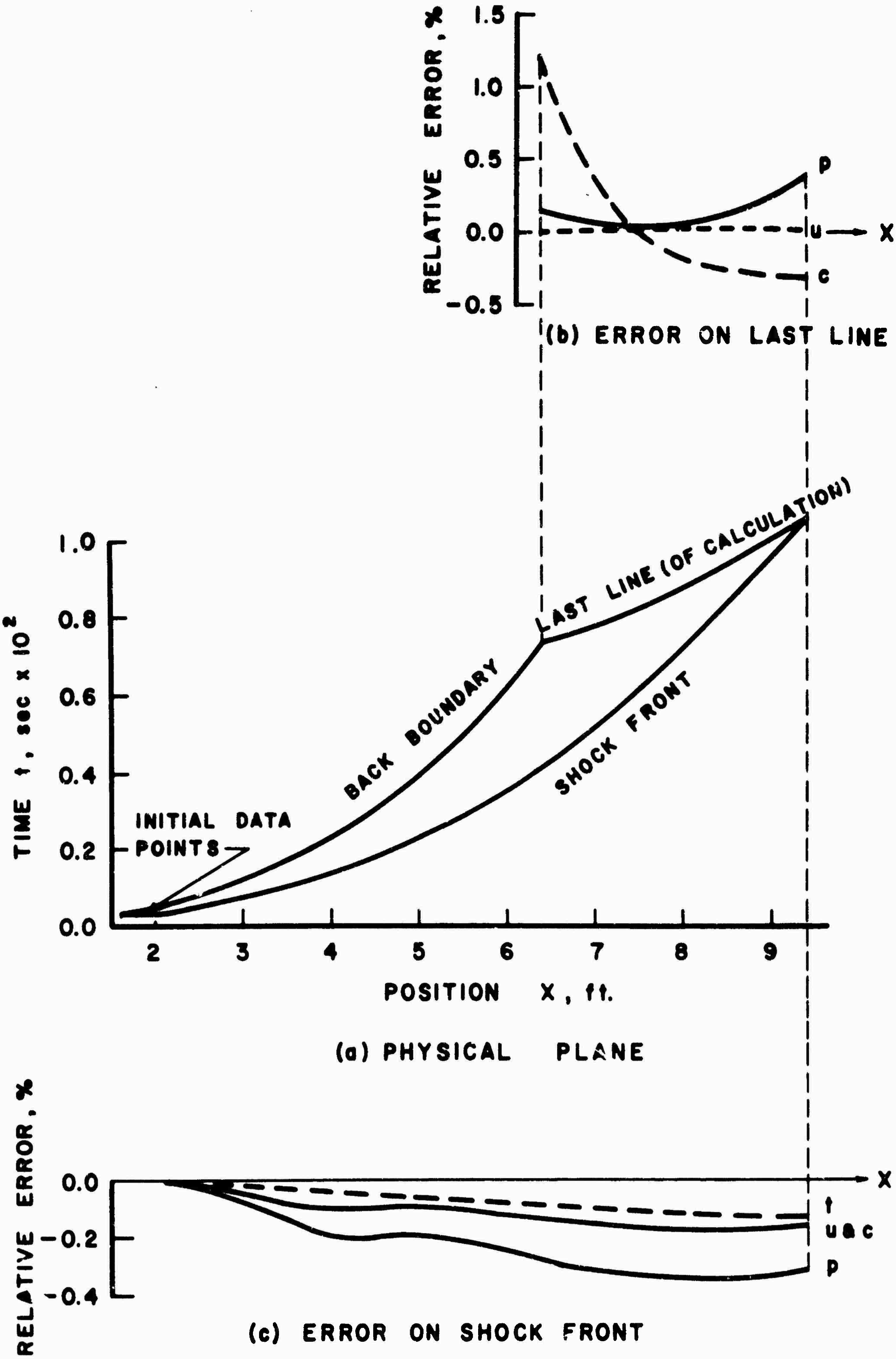

Figure 7— Spherical Wave Error Analysis

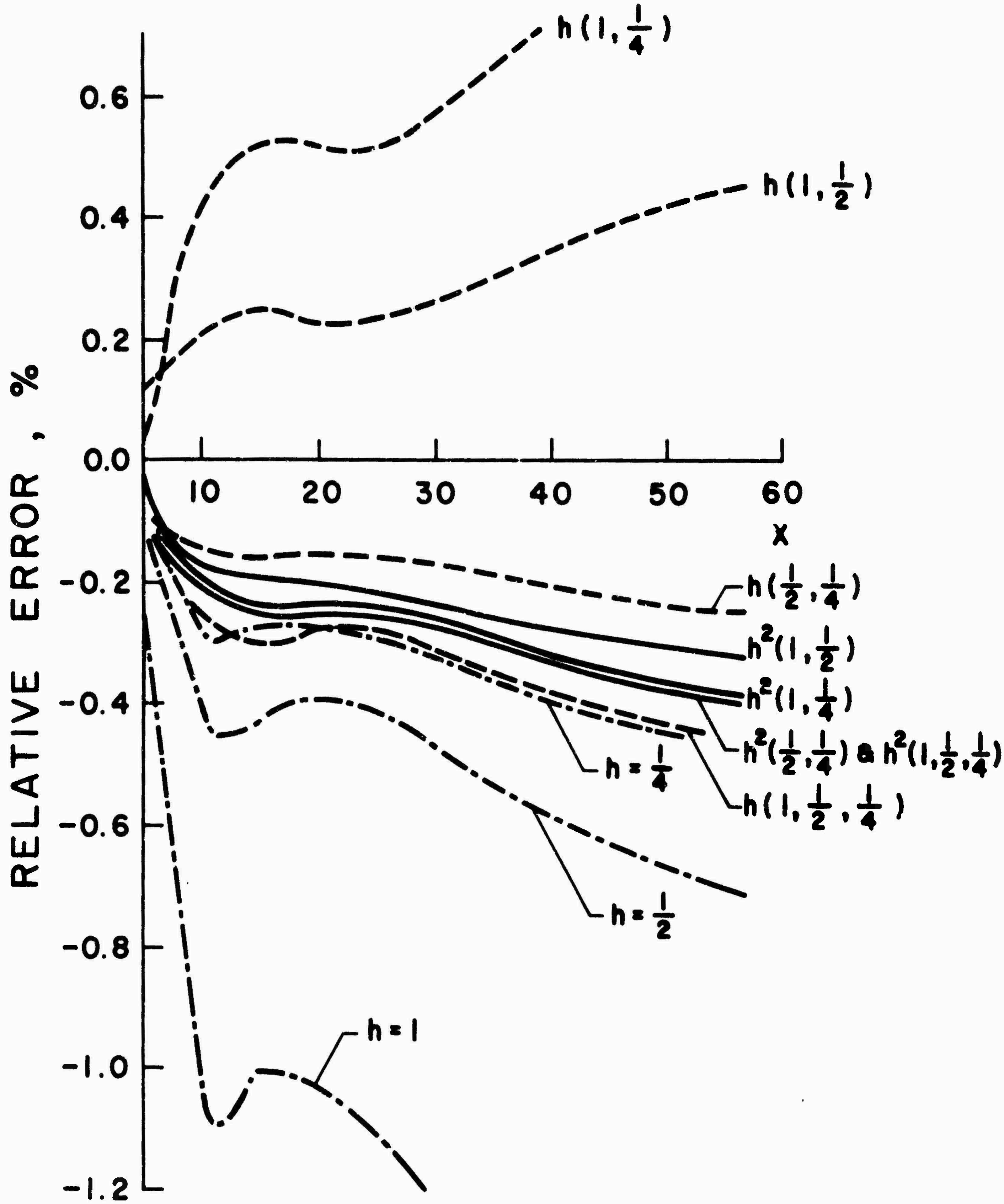

Figure 8 Curves of Relative Error in Pressure for the Plane Blast Wave (k = .5)

—·—·—·— results of numerical calculation using grid sizes h = 1, h = 1/2, and h = 1/4

- - - - - - - h-type extrapolation

———————— h^2-type extrapolation

h(1, 1/4) indicates h-type extrapolation of the two results obtained from grid sizes h = 1, and h = 1/4.

APPENDIX A

Derivation of Characteristic Equations

In this appendix, the application of the method of characteristics to the one-dimensional flow of an ideal gas with area change, friction, and heat transfer[5] is presented. The governing equations are the continuity equation

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} + \frac{\rho u}{A} \frac{dA}{dx} = 0 \tag{A.1}$$

the momentum equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} + F = 0 \tag{A.2}$$

and the energy equation

$$(\gamma - 1)\rho(q + uF) = \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} - c^2 \left(\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} \right) \tag{A.3}$$

where F is the wall-friction term and q represents the heat transfer rate. From the ideal gas equation of state and the definition of sound velocity, we have

$$c^2 = \gamma \frac{p}{\rho} \tag{A.4}$$

With this equation, the properties u, p, and ρ may be treated as dependent variables which are governed by equations (A.1), (A.2), and (A.3). For regions where these variables are continuous, three equations for the total differentials du, dp, and dρ may be written. Combining these differential relations with the governing equations, (A.1) to (A.3), we obtain the following set of six equations which may be solved for the

six partial derivatives of u, p, and ρ.

$$
\begin{aligned}
&+dx\frac{\partial u}{\partial x}+dt\frac{\partial u}{\partial t} &&= du\\
&+dx\frac{\partial p}{\partial x}+dt\frac{\partial p}{\partial t} &&= dp\\
&+dx\frac{\partial \rho}{\partial x}+dt\frac{\partial \rho}{\partial t} &&= d\rho\\
&+\rho\frac{\partial u}{\partial x}+u\frac{\partial \rho}{\partial x}+\frac{\partial \rho}{\partial t} &&= -\frac{\rho u}{A}\frac{dA}{dx}\\
&+u\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}+\frac{1}{\rho}\frac{\partial p}{\partial x} &&= -F\\
&+u\frac{\partial p}{\partial x}+\frac{\partial p}{\partial t}-uc^2\frac{\partial \rho}{\partial x}-c^2\frac{\partial \rho}{\partial t} &&= (\gamma-1)\rho(uF+q)
\end{aligned}
\tag{A.5}
$$

Solving these equations for ∂u/∂x by Cramer's Rule, we obtain

$$\frac{\partial u}{\partial x}=\frac{N}{D} \tag{A.6}$$

where

$$
D=\begin{vmatrix}
dx & dt & 0 & 0 & 0 & 0\\
0 & 0 & dx & dt & 0 & 0\\
0 & 0 & 0 & 0 & dx & dt\\
\rho & 0 & 0 & 0 & u & 1\\
u & 1 & 1/\rho & 0 & 0 & 0\\
0 & 0 & u & 1 & -uc^2 & -c^2
\end{vmatrix}
=(u\,dt-dx)[(u+c)dt-dx][(u-c)dt-dx]
$$

and

$$
N=\begin{vmatrix}
du & dt & 0 & 0 & 0 & 0\\
dp & 0 & dx & dt & 0 & 0\\
d\rho & 0 & 0 & 0 & dx & dt\\
-\rho u/A\ dA/dx & 0 & 0 & 0 & u & 1\\
-F & 1 & 1/\rho & 0 & 0 & 0\\
(\gamma-1)\rho(uF+q) & 0 & u & 1 & -uc^2 & -c^2
\end{vmatrix}
$$

$$
=(u\,dt-dx)\Big[\frac{u}{A}\frac{dA}{dx}c^2(dt)^2-(\gamma-1)(uF+q)(dt)^2-(F\,dt+du)(u\,dt-dx)
+\frac{dp}{\rho}dt\Big]
$$

The vanishing of D produces the three physical characteristics

I-characteristics $\quad \frac{dx}{dt} = u + c \quad$ (A.7)

II-characteristics $\quad \frac{dx}{dt} = u - c \quad$ (A.8)

III-characteristics $\quad \frac{dx}{dt} = u \quad$ (A.9)

Notice that the III-characteristics are also path lines. Along these physical characteristics, the derivative $\partial u/\partial x$ is indeterminate and therefore may be discontinuous. To insure that $\partial u/\partial x$ is indeter- and not infinite, the numerator, N, of equation (A.6) must also vanish along the characteristics. Along the III-characteristic, $(udt-dx) = 0$, and N vanishes identically. Along the I and II-characteristic, $dx/dt = u \pm c$, and the vanishing of N produces the following state characteristics (or compatibility equations).

$$du = \mp \frac{uc}{A}\frac{dA}{dx}dt \mp \frac{dp}{c\rho} \pm (\gamma-1)\frac{q}{c}dt - F\left[1 \mp (\gamma-1)\frac{u}{c}\right]dt \qquad \text{(A.10)}$$

The upper signs refer to I-characteristics and the lower signs refer to II-characteristics.

When equations (A.5) are solved for $\partial u/\partial t$, $\partial p/\partial x$, and $\partial p/\partial t$, the vanishing of the numerators yields results which are identical to those for $\partial u/\partial x$. When equations (A.5) are solved for $\partial \rho/\partial x$ and $\partial \rho/\partial t$, the numerators of these solutions do not contain the common factor (udt-dx). The vanishing of these numerators yields, in addition to equation (A.10), the III-state characteristic,

$$d\rho = \frac{1}{c^2}\left[dp - (\gamma-1)\rho(uF+q)dt\right] \qquad \text{(A.11)}$$

The solutions of equations (A.5) for the other five derivatives are given below for reference.

$$\frac{\partial u}{\partial t} = \frac{1}{D}(u\,dt - dx)\Big\{\Big[(\gamma-1)(uF+q) - \frac{uc^2}{A}\frac{dA}{dx}\Big]dx\,dt + \Big[F(u\,dt-dx) - u\,du - \frac{dp}{\rho}\Big]dx + (u^2-c^2)\,du\,dt\Big\}$$

$$\frac{\partial p}{\partial x} = \frac{1}{D}(u\,dt - dx)\Big\{\Big[(\gamma-1)\rho(uF+q)\,dt - \frac{\rho uc^2}{A}\frac{dA}{dx}dt - dp\Big](u\,dt-dx) + \rho c^2\,dt\,(F\,dt + du)\Big\}$$

$$\frac{\partial p}{\partial t} = \frac{1}{D}(u\,dt - dx)\Big\{\Big[\frac{\rho uc^2}{A}\frac{dA}{dx}dx - (\gamma-1)\rho(uF+q)\,dx + u\,dp\Big]\times (u\,dt-dx) - \rho c^2 F\,dx\,dt - c^2\,dp\,dt - \rho c^2\,du\,dx\Big\} \tag{A.12}$$

$$\frac{\partial \rho}{\partial x} = \frac{1}{D}\Big\langle(u\,dt - dx)\Big\{\Big(-\frac{\rho u}{A}\frac{dA}{dx}dt - d\rho\Big)(u\,dt-dx) + \rho F(dt)^2 + \rho\,du\,dt\Big\} + (\gamma-1)\rho(uF+q)(dt)^3 + d\rho(dt)^2c^2 - dp(dt)^2\Big\rangle$$

$$\frac{\partial \rho}{\partial t} = \frac{1}{D}\Big\langle(u\,dt - dx)\Big\{\Big(\frac{\rho u}{A}\frac{dA}{dx}dx + u\,d\rho\Big)(u\,dt-dx) - \rho\,dx(F+du)\Big\} - (\gamma-1)\rho(uF+q)\,dx(dt)^2 + dp\,dx\,dt - uc^2\,d\rho(dt)^2\Big\rangle$$

For frictionless flow without heat transfer, both F and q vanish. The quantity (1/A)(dA/dx) can be evaluated for the following specific types of one-dimensional motion:

1) Plane motion $\quad A = \text{constant}, \quad \frac{1}{A}\frac{dA}{dx} = 0$

2) Cylindrical motion $\quad A = 2\pi x\ell, \quad \frac{1}{A}\frac{dA}{dx} = \frac{1}{x}$

3) Spherical motion $\quad A = 4\pi x^2, \quad \frac{1}{A}\frac{dA}{dx} = \frac{2}{x}$

where x represents the radial coordinate for the cylindrical and spherical cases. These results may be summarized by writing

$$\frac{1}{A}\frac{dA}{dx} = \frac{\nu-1}{x} \tag{A.13}$$

where $\nu = 1$, 2, or 3 corresponds to plane, cylindrical, or spherical motions, respectively. The physical and state characteristic equations for one-dimensional flow of an ideal gas without friction and heat transfer become

Physical Characteristics | State Characteristics

$$\left(\frac{dx}{dt}\right)_I = u + c \qquad (du)_I = -\frac{(dp)_I}{\rho c} - (\nu-1)\frac{uc}{x}(dt)_I$$

$$\left(\frac{dx}{dt}\right)_{II} = u - c \qquad (du)_{II} = \frac{(dp)_{II}}{\rho c} + (\nu-1)\frac{uc}{x}(dt)_{II} \tag{A.14}$$

$$\left(\frac{dx}{dt}\right)_{III} = u \qquad (d\rho)_{III} = \frac{1}{c^2}(dp)_{III}$$

It is convenient to eliminate the density ρ from the characteristic equations by using equation (A.4). When this substitution is made, and the third state characteristic equation integrated, the following characteristic equations, in terms of p, u, and c, are obtained.

Physical Characteristics | State Characteristics

$$\left(\frac{dx}{dt}\right)_{I,II} = u \pm c \qquad (du)_{I,II} = \mp\frac{c}{\gamma p}(dp)_{I,II} \mp (\nu-1)\frac{uc}{x}(dt)_{I,II} \tag{A.15}$$

$$\left(\frac{dx}{dt}\right)_{III} = u \qquad \left(\frac{p}{p_0}\right)_{III} = \left(\frac{c}{c_0}\right)_{III}^{\frac{2\gamma}{\gamma-1}}$$

The constants c_o and p_o must be evaluated for each III-characteristic.

APPENDIX B

Iteration Procedure

In this appendix, the solution of the finite-difference characteristic equations (14) to (17), for the boundary conditions specified in Section IV, is described.

The basic step in the solution of unsteady flow problems by the method of characteristics is the determination of all variables, u, c, p, x, and t, at a new point when these variables are known at two existing points. Repeated application of this step, using suitable initial and boundary conditions, produces the solution of the problem. For the blast wave problem, three different types of points exist, each requiring a different procedure in the basic step. These three types of points are

1) interior points, which lie within the region of numerical solution,

2) points on the shock front, and

3) points on the back boundary.

The three procedures, which were used to solve for the three types of points, are outlined below.

1) Interior Points

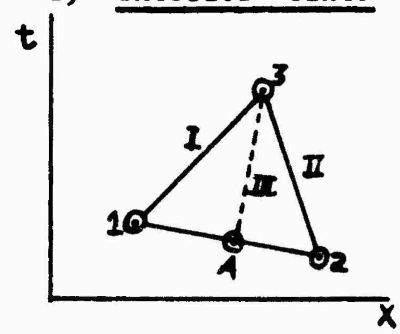

Figure B.1 Physical Plane

Referring to Figure B.1, all variables are assumed known at points 1 and 2, and the variables at point 3 are to be determined. Point A is located, in the physical plane, at the intersection of the III-characteristic (path line) passing through point 3 and the straight line joining points 1 and 2. The properties u_A, c_A, and p_A, at point A, are determined by a linear interpolation of the properties at points 1 and 2. The finite-difference forms of the physical characteristic equations are, from equations (14) to (17)

along I $$\frac{x_3 - x_1}{t_3 - t_1} = \overline{u}_{13} + \overline{c}_{13} \tag{B.1}$$

along II $$\frac{x_3 - x_2}{t_3 - t_2} = \overline{u}_{23} + \overline{c}_{23} \tag{B.2}$$

along III $$\frac{x_3 - x_A}{t_3 - t_A} = \overline{u}_{3A} \tag{B.3}$$

The corresponding state characteristics are

along I $$u_3 - u_1 = -\frac{\overline{c}_{13}}{\gamma \overline{p}_{13}}(p_3 - p_1) - (\gamma - 1)\frac{\overline{u}_{13}\overline{c}_{13}}{X_{13}}(t_3 - t_1) \tag{B.4}$$

along II $$u_3 - u_2 = \frac{\overline{c}_{23}}{\gamma \overline{p}_{23}}(p_3 - p_2) + (\gamma - 1)\frac{\overline{u}_{23}\overline{c}_{23}}{X_{23}}(t_3 - t_2) \tag{B.5}$$

along III $$\frac{p_3}{p_A} = \left(\frac{c_3}{c_A}\right)^{\frac{2\gamma}{\gamma-1}} \tag{B.6}$$

where the barred quantities represent average values, i.e.,

$$\overline{u}_{13} = \frac{1}{2}(u_1 + u_3)$$

The equation of the line between points 1 and 2 is

$$\frac{x_2 - x_1}{t_2 - t_1} = \frac{x_A - x_1}{t_A - t_1} \tag{B.7}$$

and the linear interpolation formulas, which are used to determine the properties at point A, are

$$u_A = u_1 + \frac{x_A - x_1}{x_2 - x_1}(u_2 - u_1) \tag{B.8}$$

$$c_A = c_1 + \frac{x_A - x_1}{x_2 - x_1}(c_2 - c_1) \tag{B.9}$$

$$p_A = p_1 + \frac{x_A - x_1}{x_2 - x_1}(p_2 - p_1) \tag{B.10}$$

From these ten equations, the ten unknowns, x_3, t_3, u_3, c_3, p_3, x_A, t_A, u_A, c_A, and p_A, may be determined. The solution is accomplished by the following iteration procedure.

First Iteration

A single prime is used to designate the first approximation of the variables.

Equations (B.4) and (B.5) are solved simultaneously for the properties u_3 and p_3 by neglecting terms containing t and using the properties at point 1 to replace the average properties between 1 and 3, and the properties at 2 to replace the average properties between 2 and 3. These values are then designated by u_3' and p_3'. An estimated value of c_3, which is called c_3^o, is calculated from the equation

$$c_3^o = c_1 \left(\frac{p_3'}{p_1}\right)^{\frac{\gamma-1}{2\gamma}}$$

Next, equations (B.1) and (B.2) are solved simultaneously for x_3' and t_3' by using the previously determined values of c_3^o and u_3'. Equations (B.3) and (B.7) are then solved simultaneously for x_A' and t_A' by using u_3' in place of $\bar{u}_{3A}$. The properties u_A', c_A', and p_A' are determined from the interpolation formulas, equations (B.8), (B.9), and (B.10), and the calculated value of x_A'. The quantity c_3' is calculated

by inserting the values c_A', p_A', and p_3' into equation (B.6). The first approximation of the properties at points 3 and A is now complete.

Second Iteration

Once the first approximation is established, the following procedure may be used to obtain the second and subsequent approximations of the variables at point 3. A double prime is used to designate the second approximation.

The values of u_3'' and p_3'' are calculated from the following version of equations (B.4) and (B.5),

$$u_3'' - u_1 = -\frac{\bar{c}_{13'}}{\gamma \bar{p}_{13'}}(p_3'' - p_1) - (\gamma - 1)\frac{\bar{u}_{13'}\bar{c}_{13'}}{x_{13'}}(t_3' - t_1)$$

$$u_3'' - u_2 = \frac{\bar{c}_{23'}}{\gamma \bar{p}_{23'}}(p_3'' - p_1) + (\gamma - 1)\frac{\bar{u}_{23'}\bar{c}_{23'}}{x_{23'}}(t_3' - t_1)$$

where $\bar{c}_{13'} = 1/2(c_1 + c_3')$, etc. Equations (B.1) and (B.2) are then solved simultaneously for the new position of point 3, x_3'', and t_3'', by using the most recent values u_3'' and c_3'. Similarly, x_A'' and t_A'' are calculated from equations (B.3) and (B.7). Again, the most recently calculated values of x_3'', t_3'', u_A', and u_3'', are used. With x_A'' established, the interpolation formulas, equations (B.8) to (B.10), are used to calculate u_A'', c_A'', and p_A''. Finally, the quantity c_3'' is calculated from equation (B.6) by using the values c_A'', p_A'', and p_3''.

The second approximation of the properties at points 3 and A is now complete. Repeating the procedures in the second iteration, with double primed quantities instead of single primed quantities used as input, we can obtain the third approximation. This iteration process may be continued until the desired degree of accuracy is reached. In

the present calculation, the coefficients in equations (B.4) and (B.5),

$$\left(\frac{\bar{C}_{13}}{P_{13}}\right), \left(\frac{\bar{U}_{13}\bar{C}_{13}}{X_{13}}\right), \left(\frac{\bar{C}_{23}}{P_{23}}\right), \text{ and } \left(\frac{\bar{U}_{23}\bar{C}_{23}}{X_{23}}\right)$$

were used to determine the convergence of the variables at point 3. The iteration was stopped when each of the above four coefficients converged to within .005% of its value in the previous iteration. When this iteration procedure is used, the variables x_3 and t_3 converge faster than the properties u_3, c_3, and p_3. For most points in this problem, not more than four iterations were required to achieve the desired accuracy.

A few other iteration schemes* have been tested and compared with the one just described. These schemes give the desired accuracy, but the one just described seems to be most rapid.

2) Points on the Shock Front

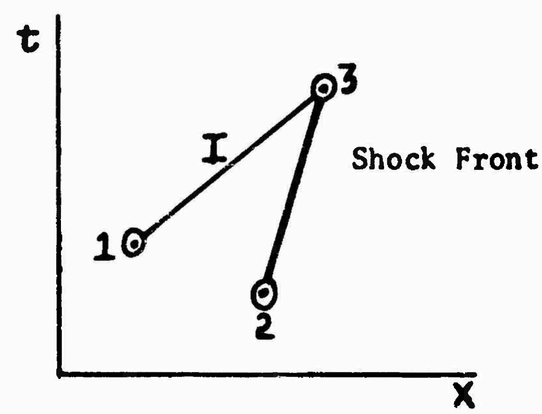

Figure B.2 Physical Plane

Referring to Figure B.2, all variables are assumed known at points 1 and 2, and the variables at point 3 are to be determined. Points 2 and 3 lie on the shock front. The finite-difference forms of the I-physical and state characteristics are, from equations (14) to (17)

*For instance, instead of solving for x_A'' and t_A'' from equations (B.3) and (B.7), we could solve equations (B.3), (B.7) and (B.8) simultaneously for x_A'', t_A'', and u_A'' by using the previously determined values x_3'', t_3'', and u_3''. But this method involves the solution of a quadratic equation which has been found impractical.

$$\frac{x_3 - x_1}{t_3 - t_1} = \bar{u}_{13} + \bar{c}_{13} \tag{B.11}$$

$$u_3 - u_1 = -\frac{\bar{c}_{13}}{\gamma \bar{p}_{13}}(p_3 - p_1) - (\nu - 1)\frac{\bar{u}_{13}\bar{c}_{13}}{\bar{x}_{13}}(t_3 - t_1) \tag{B.12}$$

From the definition of shock wave velocity, $(dx/dt)_{shock} = U$, we have

$$\frac{x_3 - x_2}{t_3 - t_2} = U_{23} \tag{B.13}$$

Also, the properties at point 3 must satisfy the strong shock equations, (11) to (13); therefore,

$$u_3 = \frac{2}{\gamma + 1} U_3 \tag{B.14}$$

$$c_3 = \frac{\sqrt{2\gamma(\gamma - 1)}}{(\gamma + 1)} U_3 \tag{B.15}$$

$$p_3 = \frac{2}{\gamma + 1} \rho_1 U_3^2 \tag{B.16}$$

From these six equations, (B.11) to (B.16), the six unknowns, x_3, t_3, u_3, c_3, p_3, and U_3, may be determined. The solution is accomplished by the following iteration procedure.

First Iteration

Equations (B.12), (B.14), and (B.16) are solved simultaneously for u_3, p_3, and U_3 by neglecting the term containing t in equation (B.12). These values are the first approximations and are designated by u_3', p_3', and U_3'. The quantity c_3' is calculated from equation (B.15), where U_3' is inserted for U_3. The position of point 3, x_3' and t_3', is found by simultaneously solving equations (B.11) and (B.13). In these equations, the previously calculated values u_3', c_3', and U_3' are used, completing the first approximation of the variables at point 3.

Second Iteration

Once the first approximation is established, the following procedure may be used to obtain the second and subsequent approximations. Equations (B.12), (B.14), and (B.16) are solved for u_3'', p_3'', and U_3'' by using some of the values obtained in the first iteration, i.e.,

$$u_3'' - u_1 = -\frac{\bar{c}_{13'}}{\gamma \bar{p}_{13'}}(p_3'' - p_1) - (\gamma - 1)\frac{\bar{u}_{13'}\bar{c}_{13'}}{\bar{x}_{13'}}(t_3' - t_1)$$

$$u_3'' = \frac{2}{\gamma + 1} U_3''$$

$$p_3'' = \frac{2}{\gamma + 1} \rho_1 (U_3'')^2$$

The quantity c_3'' is not calculated from equation (B.15) with U_3'' inserted for U_3. The position, x_3'' and t_3'', is found by simultaneously solving equations (B.11) and (B.13). Again, the most recent values u_3'', c_3'', and U_3'' are used. The second approximation of the properties at point 3 is now complete. By repeating the above procedures, with double primed quantities replacing single primed quantities, we can obtain the third approximation. This iteration process may be continued until the desired accuracy is obtained. In the present calculation, the iteration was stopped when each of the coefficients in equation (B.12), $(\bar{c}_{13}/\bar{p}_{13})$ and $(\bar{u}_{13}\bar{c}_{13}/\bar{x}_{13})$, converged to within .005% of its value in the previous iteration.

3) Points on the Back Boundary

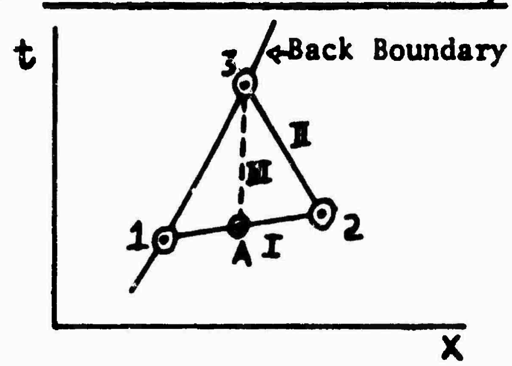

Figure B.3 Physical Plane

Referring to Figure B.3, all variables at points 1 and 2 are assumed known, and the variables at point 3 are to be determined. Points 1 and 3 are on the back boundary. Point 2 is an interior point, and points 1 and 2 are on the same I-characteristic. Point A is located at the intersection of the III-characteristic passing through point 3 and the I-characteristic joining points 1 and 2. The properties at point A will be determined by a linear interpolation of the properties at points 1 and 2. The finite-difference forms of the physical characteristics are, from equations (14) and (16)

along II $$\frac{x_3 - x_2}{t_3 - t_2} = \bar{u}_{23} - c_{23} \qquad \text{(B.17)}$$

along III $$\frac{x_3 - x_A}{t_3 - t_A} = u_{3A} \qquad \text{(B.18)}$$

The corresponding state characteristics are

along II $$u_3 - u_2 = \frac{c_{23}}{\gamma p_{23}}(p_3 - p_2) + (\nu - 1)\frac{\bar{u}_{23}\bar{c}_{23}}{x_{23}}(t_3 - t_2) \qquad \text{(B.19)}$$

along III $$\frac{p_3}{p_A} = \left(\frac{c_3}{c_A}\right)^{\frac{2\gamma}{\gamma - 1}} \qquad \text{(B.20)}$$

Since the I-characteristic is assumed to be a straight line between points 1 and 2, we have

$$\frac{x_2 - x_1}{t_2 - t_1} = \frac{x_A - x_1}{t_A - t_1} \qquad \text{(B.21)}$$

Point 3 must lie on the back boundary; therefore, it must satisfy the equation of the back boundary curve presented in section III.

$$x_3 = K\left(\frac{E}{A}\right)^{\frac{1}{2+\nu}} t_3^{\frac{2}{2+\nu}} \qquad \text{(B.22)}$$

where k = constant (the position ratio). Also, as discussed in section III, the particle velocity at points on the back boundary is prescribed by the exact solution; therefore,

$$u_3 = F\frac{5}{3(\nu+2)}\left(\frac{E}{A}\right)^{\frac{1}{2+\nu}} t_3^{-\frac{\nu}{2+\nu}} \qquad \text{(B.23)}$$

where F = constant (the velocity ratio u_b/u_s). The linear interpolation formulas, which are used to determine the properties at point A, are

$$u_A = u_1 + \frac{x_A - x_1}{x_2 - x_1}(u_2 - u_1) \qquad \text{(B.24)}$$

$$c_A = c_1 + \frac{x_A - x_1}{x_2 - x_1}(c_2 - c_1) \qquad \text{(B.25)}$$

$$p_A = p_1 + \frac{x_A - x_1}{x_2 - x_1}(p_2 - p_1) \qquad \text{(B.26)}$$

From these ten equations, (B.17) to (B.26), the ten unknowns, x_3, t_3, u_3, c_3, p_3, x_A, t_A, u_A, c_A, and p_A, may be determined. The solution is accomplished by the following iteration procedure.

First Iteration

Equations (B.17) and (B.22) are solved simultaneously for x_3 and t_3 by using u_2 for $\bar{u}_{23}$ and c_2 for $\bar{c}_{23}$. These values are then designated by x_3' and t_3', respectively. The first approximation of the particle velocity u_3' is determined by using t_3' in equation (B.23) and solving for u_3. The quantity p_3' is determined from equation (B.19) by neglecting the term containing t and using the properties at point 2 in place of the average properties. Next, x_A' and t_A' are found by solving equation (B.18) and (B.21) with $\bar{u}_{3A}$ replaced by u_3'. The properties u_A', c_A', and p_A' are determined from the interpolation formulas, equations (B.24), (B.25), and (B.26), and the calculated position x_A'. The sound speed c_3' may now be calculated from equation (B.20). The first approximation of the properties at points 3 and A is now complete.

Second Iteration

The following procedure may be used to calculate the second and subsequent approximations.

Equations (B.17) and (B.22) are solved for x_3'' and t_3'' by using the average properties $\bar{u}_{23}'$ and $\bar{c}_{23}'$. The particle velocity u_3'' is calculated from equation (B.23) by using t_3''. Next, equation (B.19) is solved for p_3'' by using some of the values obtained in the first iteration, i.e.,

$$u_3'' - u_2 = \frac{\bar{c}_{23}'}{\gamma \bar{p}_{23}'}(p_3'' - p_2) + (\gamma - 1)\frac{\bar{u}_{23}'' \bar{c}_{23}'}{\bar{x}_{23}''}(t_3'' - t_2)$$

The position of point A, x_A'' and t_A'', is determined from equation (B.18) and (B.21) with $\bar{u}_{3''A'}$ used as the average particle velocity. The position of point A may also be determined by the alternate procedure described in the footnote of the interior point procedure. Next, the interpolation formulas, (B.24) to (B.26), are used to determine u_A'', c_A'', and p_A''. Then the sound speed c_3'' is calculated from equation (B.20), completing the second approximation of the properties at points 3 and A. As described in the general point iteration procedure, this process may be continued until the desired accuracy is reached. In the present calculations, the iteration was stopped when the coefficients in equation (B.19), $(\bar{c}_{23}/\bar{p}_{23})$ and $(\bar{u}_{23}\bar{c}_{23}/\bar{x}_{23})$, converged to within .005%.

APPENDIX C

Error Analysis of the Finite-Difference Equations

It is desirable to determine the nature of the truncation error which is introduced by replacing the governing differential characteristic equations (4), (5), and (6) by the finite-difference equations (14), (15), and (16). Since these equations are of the same general form, only one typical equation will be analyzed. This equation is the I-state characteristic equation (6).

$$\frac{du}{d\alpha} = -\frac{c}{\gamma p}\frac{dp}{d\alpha} - (\gamma - 1)\frac{cu}{x}\frac{dt}{dx} \qquad (C.1)$$

where α represents arc length along the I-characteristic. When this equation is expressed in finite-difference form, the coefficients c/p and cu/x may be approximated by either $\bar{c}/\bar{p}$ and $\bar{c}\bar{u}/\bar{x}$ or $\overline{(c/p)}$ and $\overline{(cu/x)}$, i.e., either by averaging the variables individually, or by averaging the coefficients which are combinations of the variables. It will be shown that the types of error involved in both cases are the same.

Referring to Figure C.1, the finite-difference characteristic equation (15) is to be written for point 2, in terms of values of the variables at points 1 and 3, both $\Delta\alpha$ distance away from 2 along the I-characteristic.

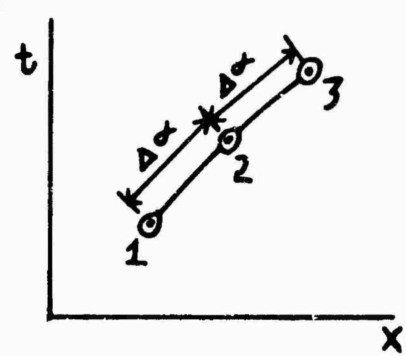

Figure C.1 Physical Plane

With individual variables averaged, eq. (15) becomes

$$\frac{\Delta u}{\Delta \alpha} = -\frac{\bar{c}_{13}}{\gamma \bar{p}_{13}} \frac{\Delta p}{\Delta \alpha} - (\gamma - 1) \frac{\bar{c}_{13} \bar{u}_{13}}{\bar{x}_{13}} \frac{\Delta t}{\Delta \alpha} \tag{C.2}$$

where Δ represents the central difference, i.e., $\Delta u = u_3 - u_1$; and the barred quantities represent averages between points 1 and 3, e.g.,

$$\bar{u}_{13} = \frac{1}{2}(u_1 + u_3)$$

The Taylor series expansions (assuming they exist) for u_3 and u_1 about point 2 are,

$$u_3 = u_2 + \left(\frac{du}{d\alpha}\right)_2 \Delta\alpha + \frac{1}{2!}\left(\frac{d^2u}{d\alpha^2}\right)_2 \Delta\alpha^2 + \cdots$$

$$u_1 = u_2 - \left(\frac{du}{d\alpha}\right)_2 \Delta\alpha + \frac{1}{2!}\left(\frac{d^2u}{d\alpha^2}\right)_2 \Delta\alpha^2 + \cdots$$

The other variables at points 1 and 3 can be expanded in a similar manner. By adding and subtracting the above series expansions, and rearranging, we obtain

$$u_2 = \bar{u}_{13} - \frac{1}{2!}\left(\frac{d^2u}{d\alpha^2}\right) \Delta\alpha^2 + \cdots$$

$$\left(\frac{du}{d\alpha}\right)_2 = \frac{\Delta u}{2\Delta\alpha} - \frac{1}{3!}\left(\frac{d^3u}{d\alpha^3}\right) \Delta\alpha^2 - \cdots$$

Again, similar expressions for c_2, p_2, x_2, $(dp/d\alpha)_2$ and $(dt/d\alpha)_2$ can be formed from the corresponding series expansions.

Substituting these expressions into the exact differential equation (C.1), which is valid at point 2, we obtain

$$\frac{\Delta u}{\Delta \alpha} + \frac{\bar{c}_{13}}{\gamma \bar{p}_{13}} \frac{\Delta p}{\Delta \alpha} + (\gamma - 1) \frac{\bar{c}_{13} \bar{u}_{13}}{\bar{x}_{13}} \frac{\Delta t}{\Delta \alpha} = N_1(\Delta\alpha)^2 + N_2(\Delta\alpha)^4 + \cdots \tag{C.3}$$

where N_1, N_2, etc., are independent of $\Delta\alpha$ and depend only upon the position of point 2. By comparing eq. (C.3) with (C.2), we observe that the terms $N_1(\Delta c)^2 + N_2(\Delta\alpha)^4 + \ldots$ are neglected in eq. (C.2). Therefore, the error e introduced by the replacement of eq. (C.1) by (C.2) is of the order $(\Delta\alpha)^2$, or

$$e = N_1(\Delta\alpha)^2 + N_2(\Delta\alpha)^4 + \cdots \tag{C.4}$$

With coefficients averaged, equation (15) becomes

$$\frac{\Delta u}{\Delta\alpha} = -\frac{1}{\gamma}\overline{\left(\frac{c}{p}\right)}_{13}\frac{\Delta p}{\Delta\alpha} - (\gamma - 1)\overline{\left(\frac{cu}{x}\right)}_{13}\frac{\Delta t}{\Delta\alpha} \tag{C.5}$$

where

$$\overline{\left(\frac{c}{p}\right)}_{13} = \frac{1}{2}\left[\frac{c_1}{p_1} + \frac{c_3}{p_3}\right] \text{ and } \overline{\left(\frac{cu}{x}\right)}_{13} = \frac{1}{2}\left[\frac{c_1u_1}{x_1} + \frac{c_3u_3}{x_3}\right]$$

For this case, the quantities c/p, cu/x, u, p, and t at points 3 and 1 are expanded into Taylor series about point 2.

$$\frac{c_3}{p_3} = \left(\frac{c}{p}\right)_3 = \left(\frac{c}{p}\right)_2 + \frac{d}{d\alpha}\left(\frac{c}{p}\right)_2\Delta\alpha + \frac{1}{2!}\frac{d^2}{d\alpha^2}\left(\frac{c}{p}\right)_2\Delta\alpha^2 + \cdots$$

$$\frac{c_1}{p_1} = \left(\frac{c}{p}\right)_1 = \left(\frac{c}{p}\right)_2 - \frac{d}{d\alpha}\left(\frac{c}{p}\right)_2\Delta\alpha + \frac{1}{2!}\frac{d^2}{d\alpha^2}\left(\frac{c}{p}\right)_2\Delta\alpha^2 + \cdots$$

By combining these two equations, we have

$$\left(\frac{c}{p}\right)_2 = \overline{\left(\frac{c}{p}\right)}_{13} - \frac{1}{2!}\frac{d^2}{d\alpha^2}\overline{\left(\frac{c}{p}\right)}_2\Delta\alpha^2 - \frac{1}{4!}\frac{d^4}{d\alpha^4}\left(\frac{c}{p}\right)_2\Delta\alpha^4$$

If we substitute the above equation and similar expressions for $(dp/d\alpha)_2$, $(cu/x)_2$ and $(dt/d\alpha)_2$ into the exact equation (C.1), we obtain

$$\frac{\Delta u}{\Delta\alpha} + \left(\frac{c}{\gamma p}\right)_{13}\frac{\Delta p}{\Delta\alpha} + (\gamma - 1)\overline{\left(\frac{cu}{x}\right)}_{13}\frac{\Delta t}{\Delta\alpha} = K_1(\Delta\alpha)^2 + K_2(\Delta\alpha)^4 + \cdots \tag{C.6}$$

where K_1, K_2, etc., are independent of $\Delta\alpha$. Comparing eq. (C.6) with eq. (C.5), it is observed that the terms $K_1(\Delta\alpha)^2 + K_2(\Delta\alpha)^4 + \ldots$ are neglected in eq. (C.5). Therefore, the replacement of (C.1) by (C.5) introduces an error e' of order $(\Delta\alpha)^2$.

$$e' = K_1(\Delta\alpha)^2 + K_2(\Delta\alpha)^4 + \cdots$$

We conclude that for either of the two methods of averaging, the errors involved in all the finite-difference equations are of the $(\Delta\alpha)^2$-type. If h is a quantity which is in some way proportional to the arc length $\Delta\alpha$, then these errors are of the h^2 -type.

APPENDIX D

Similarity Solution of the Blast Wave Problem

In this appendix, Sedov's solution for the wave generated by the instantaneous release of a finite amount of energy is summarized (see ref. (13), chap. IV).

From dimensional considerations, the dependent variables, u, ρ, and p, may be expressed as follows:

$$u = \frac{x}{t} V(\lambda) \tag{D.1}$$

$$\rho = \frac{a}{x^{K+3} t^{s}} R(\lambda) \tag{D.2}$$

$$p = \frac{a}{x^{K+1} t^{s+2}} P(\lambda) \tag{D.3}$$

where

$$\lambda = \frac{x}{b^{1/m} t^{\delta}} \tag{D.4}$$

is a nondimensional variable and a, K, s, b, m, and δ are constants. When equations (D.1), (D.2), and (D.3) are substituted into the equations governing the one-dimensional flow of a perfect gas without heat transfer and friction, equations (1), (2), and (3), the following equations result,

$$\frac{dz}{dV} = \frac{z\{[2(V-1)+\nu(\gamma-1)V](V-\delta)^2-(\gamma-1)V(V-1)(V-\delta)-[2(V-1)+f(\gamma-1)]z\}}{(V-\delta)[V(V-1)(V-\delta)+(f-\nu V)z]} \tag{D.5}$$

$$\frac{d\ln\lambda}{dV} = \frac{z-(V-\delta)^2}{V(V-1)(V-\delta)+(f-\nu V)z} \tag{D.6}$$

$$(V-\delta)\frac{d\ln R}{d\ln\lambda} = [s+(f-\nu+3)V] - \frac{V(V-1)(V-\delta)+(f-\nu V)z}{z-(V-\delta)^2} \tag{D.7}$$

where

$$f = \frac{s+2+\delta(K+1)}{\gamma}$$

These equations are ordinary differential equations and motions corresponding to them are called self-similar. In these equations, a new variable $Z = \gamma(P(\lambda)/R(\lambda))$ has been introduced and the expression for λ, equation (D.4), has been used.

In the blast wave problem, $a = \rho_1$ and $b = E/\rho_1$, where ρ_1 is the initial density of the gas outside the wave and E is a constant which is proportional to the energy released. The exponents δ and m are determined from the dimensions of b,

$$\delta = \frac{2}{\nu+2} \qquad\qquad m = \nu + 2$$

Also, the constants $K = -3$ and $S = 0$ are determined from the dimension of ρ_1. At the shock front, $x = x_s$ and $\lambda = \lambda_s$; therefore, from equation (D.4), the equation of the shock front becomes

$$x_s = \left(\frac{E}{\rho_1}\right)^{\frac{1}{\nu+2}} t^{\frac{2}{\nu+2}} \tag{D.8}$$

where λ_s has been arbitrarily set equal to unity at the shock front.

At the shock front, the variables u, p, and ρ must satisfy the strong shock equations, which are, in terms of the new variables.

$$V(1) = \frac{4}{(\gamma+1)(\nu+2)}, \quad R(1) = \frac{\gamma+1}{\gamma-1}, \quad Z(1) = \frac{8\gamma(\gamma-1)}{(\gamma+1)^2(\nu+2)^2} \tag{D.9}$$

These strong shock boundary conditions and the stipulation that the energy within the wave zone remains constant determine a unique solution. With the above information established, the properties at the shock front become

$$u_s = \frac{4}{(\nu+2)(\gamma+1)}\left(\frac{E}{\rho_1}\right)^{\frac{1}{2}} x_s^{-\frac{\nu}{2}} \tag{D.10}$$

$$\rho_s = \frac{\gamma+1}{\gamma-1}\rho_1 \tag{D.11}$$

$$p_s = \frac{8E}{(\nu+2)^2(\gamma+1)} x_s^{-\nu} \tag{D.12}$$

Sedov obtained a solution of the differential equations (D.5), (D.6), and (D.7) which satisfies the boundary conditions (D.9). This solution, when transformed back to dimensional variables, may be expressed in terms of a parameter V as follows, (see ref. 13, page 219).

$$\frac{x}{x_s} = \left[\frac{(\nu+2)(\gamma+1)}{4} V\right]^{-\frac{2}{\nu+2}} \left[\frac{\gamma+1}{\gamma-1}\left(\frac{(\nu+2)}{2}\gamma V - 1\right)\right]^{-\alpha_2} \left[\frac{(\nu+2)(\gamma+1)}{(\nu+2)(\gamma+1)-2[2+\nu(\gamma-1)]}\left(1-\frac{2+\nu(\gamma-1)}{2}V\right)\right]^{\alpha_1} \tag{D.13}$$

$$\frac{u}{u_s} = \frac{(\nu+2)(\gamma+1)}{4} V \frac{x}{x_s} \tag{D.14}$$

$$\frac{\rho}{\rho_s} = \left[\frac{\gamma+1}{\gamma-1}\left(\frac{\nu+2}{2}\gamma V - 1\right)\right]^{\alpha_3} \left[\frac{\gamma+1}{\gamma-1}\left(1-\frac{\nu+2}{2}V\right)\right]^{\alpha_5} \left[\frac{(\nu+2)(\gamma+1)}{(\nu+2)(\gamma+1)-2[2+\nu(\gamma-1)]}\left(1-\frac{2+\nu(\gamma-1)}{2}V\right)\right]^{\alpha_4} \tag{D.15}$$

$$\frac{p}{p_s} = \left[\frac{(\nu+2)(\gamma+1)}{4} V\right]^{\frac{2\nu}{\nu+2}} \left[\frac{\gamma+1}{\gamma-1}\left(1-\frac{\nu+2}{2}V\right)\right]^{\alpha_5+1} \left[\frac{(\nu+2)(\gamma+1)}{(\nu+2)(\gamma+1)-2[2+\nu(\gamma+1)]}\left(1-\frac{2+\nu(\gamma-1)}{2}V\right)\right]^{\alpha_4-2\alpha_1} \tag{D.16}$$

where

$$\alpha_1 = \frac{(\nu+2)\gamma}{2+\nu(\gamma-1)}\left[\frac{2\nu(2-\gamma)}{\gamma(\nu+2)^2} - \alpha_2\right] \qquad \alpha_2 = \frac{1-\gamma}{2(\gamma-1)+\nu}$$

$$\alpha_3 = \frac{\nu}{2(\gamma-1)+\nu} \qquad \alpha_4 = \frac{\alpha_1(\nu+2)}{2-\gamma} \qquad \alpha_5 = \frac{2}{\gamma-2}$$

The parameter V has the range

$$\frac{2}{(\nu+2)\gamma} \leqslant V \leqslant \frac{4}{(\nu+2)(\gamma+1)}$$

for $1 < \gamma < 7$.

The complete solution of the blast wave problem is given by equations (D.10) through (D.16) together with the equation of the shock front (D.8).

TABLE II. BLAST WAVE PROPERTIES FOR PLANE, CYLINDRICAL, AND SPHERICAL WAVES WITH γ = 7/5 and 5/3

a. Plane Wave (ν=1), γ = 7/5

x/x_s	u/u_s	ρ/ρ_s	p/p_s	c/c_s
1.00000000	1.00000000	1.00000000	1.00000000	1.000000
.99000008	.98507586	.92860073	.95681441	1.015077
.98000006	.97030586	.86384121	.91702569	1.030323
.97000005	.95569410	.80496880	.88031874	1.045756
.96000003	.94124416	.75132872	.84641294	1.061392
.94999993	.92695898	.70234947	.81505749	1.077252
.94000007	.91284169	.65753374	.78602963	1.093353
.93000011	.89889401	.61644331	.75912832	1.109713
.92000017	.88511779	.57869431	.73417377	1.126353
.91000011	.87151407	.54394807	.71100374	1.143292
.90000013	.85808381	.51190647	.68947255	1.160548
.89000010	.84482706	.48230480	.66944805	1.178142
.88000013	.83174370	.45490900	.65081098	1.196093
.87000010	.81883284	.42951019	.63345260	1.214423
.86000015	.80609347	.40592327	.61727472	1.233153
.85000011	.79352378	.38398229	.60218737	1.252305
.83999997	.78112174	.36353930	.58810889	1.271900
.83000010	.76888539	.34446246	.57496507	1.291962
.82000017	.75681160	.32663233	.56268731	1.312514
.81000006	.74489716	.30994204	.55121304	1.333581
.80000021	.73313927	.29429640	.54048565	1.355188
.79000022	.72153383	.27960816	.53045204	1.377362
.78000015	.71007714	.26579956	.52106416	1.400130
.77000018	.69876538	.25280027	.51227773	1.423521
.76000011	.68759415	.24054636	.50405174	1.447565
.75000007	.67655934	.22898031	.49634869	1.472293
.74000015	.66565660	.21804985	.48913387	1.497738
.73000019	.65488133	.20770733	.48237509	1.523935
.72000025	.64422905	.19790966	.47604272	1.550920
.71000019	.63369507	.18861748	.47010924	1.578732
.70000014	.62327496	.17979508	.46454921	1.607411
.69000012	.61296423	.17141004	.45933918	1.636999
.68000009	.60275827	.16343250	.45445718	1.667543
.67000008	.59265269	.15583539	.44988294	1.699091
.66000037	.58264331	.14859405	.44559759	1.731692
.65000005	.57272490	.14168517	.44158317	1.765403
.64000030	.56289435	.13508878	.43782378	1.800279
.63000038	.55314676	.12878528	.43430378	1.836384
.62000015	.54347792	.12275718	.43100834	1.873784
.61000030	.53388451	.11698885	.42792592	1.912546
.60000031	.52436215	.11146524	.42504223	1.952748

TABLE II. a. (continued)

x/x_s	u/u_s	ρ/ρ_s	p/p_s	c/c_s
.59000036	.51490728	.10617291	.42234623	1.994469
.58000030	.50551608	.10109933	.41982682	2.037797
.57000026	.49618517	.09623310	.41747380	2.082822
.56000061	.48691149	.09156385	.41527757	2.129644
.55000062	.47769108	.08708148	.41322885	2.178373
.54000071	.46852116	.08277720	.41131921	2.229123
.53000042	.45939827	.07864253	.40954047	2.282021
.52000038	.45032008	.07467007	.40788517	2.337198
.51000059	.44128379	.07085273	.40634611	2.394802
.50000069	.43228639	.06718380	.40491642	2.454993
.49000142	.42332598	.06365748	.40358984	2.517939
.48000121	.41439873	.06026761	.40236012	2.583839
.47000134	.40550343	.05700940	.40122161	2.652887
.46000207	.39663810	.05387808	.40015893	2.725308
.45000199	.38779939	.05086870	.39919632	2.801355
.44000171	.37898584	.04797728	.39830041	2.881293
.43000155	.37019584	.04520000	.39747510	2.965416
.42000147	.36142753	.04253319	.39671643	3.054049
.41000221	.35267990	.03997365	.39602033	3.147544
.40000337	.34395096	.03751804	.39538270	3.246300
.39000176	.33523643	.03516262	.39479963	3.350793
.38000264	.32653950	.03290585	.39426782	3.461458
.37000275	.31785596	.03074428	.39378367	3.578875
.36000425	.30918647	.02867588	.39334396	3.703631
.35000328	.30052648	.02669751	.39294554	3.836461
.34000270	.29187742	.02480748	.39258553	3.978100
.33000476	.28324017	.02300405	.39226123	4.129386
.32000414	.27460923	.02128420	.39196980	4.291385
.31000672	.26598873	.01964698	.39170892	4.465126
.30000857	.25737460	.01808980	.39147606	4.651956
.29000825	.24876474	.01661065	.39126895	4.853379
.28001068	.24016272	.01520848	.39108557	5.070992
.27000359	.23155736	.01387990	.39092372	5.307044
.26001251	.22297004	.01262654	.39078182	5.563203
.25001419	.21438023	.01144371	.39065776	5.842714
.24001929	.20579662	.01033109	.39054926	6.148442
.23001775	.19721014	.00928595	.39045677	6.484449
.22002668	.18863509	.00830843	.39037681	6.854603
.2100178[illegible]	.18004683	.00739421	.39030857	7.265372
.20001246	.17146339	.00654376	.39025078	7.722500

TABLE II. BLAST WAVE PROPERTIES FOR PLANE, CYLINDRICAL, AND SPHERICAL WAVES WITH γ = 7/5 and 5/3

b. Cylindrical Wave (ν=2), γ = 7/5

x/x_s	u/u_s	ρ/ρ_s	p/p_s	c/c_s
1.00000000	1.00000000	1.00000000	1.00000000	1.000000
.99000023	.98187036	.87082494	.92731237	1.031923
.98000020	.96415348	.76311560	.86460582	1.064422
.97000012	.94685728	.67256243	.81022513	1.097580
.96000015	.92998646	.59584882	.76283978	1.131484
.95000013	.91354205	.53039435	.72135991	1.166217
.94000030	.89752275	.47417499	.68493403	1.201863
.93000022	.88192341	.42558459	.65280404	1.238507
.92000016	.86673740	.38334339	.62437809	1.276232
.91000027	.85195567	.34642050	.59915324	1.315125
.90000027	.83756658	.31397892	.57670605	1.355273
.89000032	.82355753	.28533698	.55668014	1.396766
.88000021	.80991407	.25993386	.53877216	1.439697
.87000035	.79662161	.23730774	.52272448	1.484159
.86000028	.78366368	.21707312	.50831513	1.530253
.85000040	.77102453	.19890982	.49535434	1.578083
.84000030	.75868710	.18254765	.48367726	1.627757
.83000041	.74663534	.16776004	.47314185	1.679389
.82000034	.73485240	.15435402	.46362381	1.733101
.81000022	.72332229	.14216592	.45501505	1.789019
.80000037	.71202963	.13105615	.44722103	1.847278
.79000037	.70095867	.12090413	.44015816	1.908023
.78000036	.69009484	.11160654	.43375317	1.971407
.77000041	.67942411	.10307392	.42794114	2.037593
.76000037	.66893280	.09522847	.42266452	2.106757
.75000063	.65860849	.08800280	.41787236	2.179081
.74000034	.64843811	.08133709	.41351864	2.254775
.73000047	.63841113	.07518002	.40956300	2.334044
.72000019	.62851581	.06948532	.40596864	2.417128
.71000026	.61874270	.06421298	.40270295	2.504269
.70000058	.60908208	.05932704	.39973646	2.595736
.69000115	.59952492	.05479554	.39704233	2.691818
.68000101	.59006189	.05058956	.39459632	2.792839
.67000119	.58068612	.04668403	.39237678	2.899128
.66000132	.57138992	.04305583	.39036383	3.011055
.65000129	.56216638	.03968425	.38853941	3.129020
.64000164	.55300965	.03655071	.38688719	3.253450
.63000149	.54391306	.03363795	.38539203	3.384827
.62000192	.53487216	.03093081	.38404040	3.523649
.61000184	.52588090	.02841485	.38281962	3.670492
.60000282	.51693607	.02607750	.38171845	3.825943

TABLE II. b. (continued)

x/x_s	u/u_s	ρ/ρ_s	p/p_s	c/c_s
.59000222	.50803102	.02390626	.38072608	3.990712
.58000315	.49916457	.02189082	.37983320	4.165483
.57000373	.49033143	.02002066	.37903080	4.351089
.56000345	.48152783	.01828630	.37831073	4.548428
.55000444	.47275259	.01667937	.37766569	4.758431
.54000425	.46400074	.01519136	.37708865	4.982224
.53000744	.45527377	.01381533	.37657364	5.220887
.52000608	.44656241	.01254309	.37611445	5.475927
.51000749	.43787094	.01136897	.37570614	5.748618
.50000718	.42919351	.01028617	.37534364	6.040704
.49000826	.42053110	.00928916	.37502261	6.353899
.48001011	.41188158	.00837233	.37473898	6.690230
.47000600	.40323773	.00752990	.37448878	7.052201
.46000808	.39460875	.00675797	.37426890	7.441898
.45001258	.38599023	.00605150	.37407615	7.862273
.44001434	.37737669	.00540576	.37390751	8.316748
.43001878	.36877186	.00481692	.37376046	8.808695
.42002251	.36017201	.00428085	.37363264	9.342368
.41001927	.35157103	.00379355	.37352181	9.922814
.40001952	.34297723	.00335202	.37342614	10.554770
.39002127	.33438836	.00295281	.37334385	11.244401
.38004135	.32581833	.00259333	.37327339	11.997327
.37003513	.31722839	.00226903	.37321309	12.825022
.36003655	.30864727	.00197830	.37316188	13.734168
.35005084	.30007912	.00171853	.37311859	14.734785
.34007201	.29151849	.00148699	.37308214	15.839694
.33009187	.28295811	.00128112	.37305156	17.064317
.32012595	.27441106	.00109897	.37302609	18.423639
.31018088	.26588283	.00093849	.37300503	19.936156
.30013331	.25726750	.00079598	.37298745	21.646845
.29020790	.24875756	.00067276	.37297323	23.545504
.28025050	.24022072	.00056497	.37296161	25.693058
.27026193	.23165757	.00047120	.37295222	28.133270
.26046833	.22326191	.00039179	.37294482	30.852869
.25049917	.21471602	.00032233	.37293882	34.014766
.24053268	.20617263	.00026310	.37293407	37.648559
.23059844	.19765707	.00021307	.37293043	41.835300
.22101686	.18944392	.00017233	.37292767	46.517937
.21141195	.18121087	.00013800	.37292555	51.982875
.20210235	.17323103	.00011018	.37292397	58.177597

TABLE II. BLAST WAVE PROPERTIES FOR PLANE, CYLINDRICAL, AND SPHERICAL WAVES WITH $\gamma = 7/5$ and $5/3$

c. Spherical Wave (ν=3), $\gamma = 7/5$

x/x_s	u/u_s	ρ/ρ_s	p/p_s	c/c_s
1.00000000	1.00000000	1.00000000	1.00000000	1.000000
.99000020	.97872610	.81781524	.89971703	1.048879
.98000016	.95824629	.67765852	.81849516	1.099012
.97000023	.93856766	.56797341	.75197047	1.150631
.96000031	.91968545	.48080899	.69694604	1.203963
.95000024	.90158501	.41058013	.65103994	1.259229
.94000019	.88424384	.35328933	.61244954	1.316648
.93000016	.86763231	.30602359	.57978968	1.376441
.92000028	.85171574	.26662837	.55198323	1.438830
.91000032	.83645521	.23348568	.52818167	1.504047
.90000031	.82181004	.20536700	.50771146	1.572328
.89000036	.80773849	.18132721	.49003170	1.643919
.88000052	.79419876	.16063148	.47470484	1.719081
.87000048	.78114922	.14270158	.46137315	1.798091
.86000043	.76855028	.12707988	.44974338	1.881239
.85000054	.75636395	.11339969	.43957251	1.968835
.84000052	.74455379	.10136447	.43065770	2.061214
.83000062	.73308609	.09073326	.42282925	2.158734
.82000066	.72192898	.08130799	.41594376	2.261781
.81000076	.71105318	.07292514	.40987984	2.370771
.80000076	.70043129	.06544827	.40453367	2.486155
.79000072	.69003843	.05876321	.39981652	2.608420
.78000095	.67985193	.05277360	.39565206	2.738093
.77000115	.66985054	.04739735	.39197405	2.875753
.76000103	.66001482	.04256423	.38872502	3.022030
.75000132	.65032811	.03821437	.38585510	3.177596
.74000120	.64077385	.03429527	.38332042	3.343211
.73000157	.63133853	.03076198	.38108273	3.519674
.72000170	.62200855	.02757462	.37910813	3.707889
.71000153	.61277216	.02469836	.37736680	3.908838
.70000246	.60362011	.02210287	.37583259	4.123564
.69000334	.59454187	.01976056	.37448200	4.353271
.68000320	.58552805	.01764701	.37329420	4.599282
.67000259	.57657152	.01574068	.37225086	4.863019
.66000479	.56766836	.01402270	.37133587	5.145973
.65000319	.55880658	.01247447	.37053408	5.450081
.64000410	.54998634	.01108120	.36983305	5.777094
.63000660	.54120193	.00982838	.36922099	6.129174
.62000647	.53244535	.00870257	.36868736	6.508863
.61000535	.52371418	.00769224	.36822301	6.918770
.60000740	.51500870	.00678706	.36781995	7.361676

TABLE II. c. (continued)

x/x_s	u/u_s	ρ/ρ_s	p/p_s	c/c_s
.59000839	.50632217	.00597684	.36747065	7.841068
.58001489	.49765766	.00525316	.36716878	8.360305
.57001584	.48900326	.00460709	.36690825	8.924111
.56001335	.48035874	.00403143	.36668391	9.537092
.55001672	.47173035	.00351990	.36649147	10.203909
.54002796	.46311823	.00306626	.36632683	10.930227
.53003275	.45450867	.00266420	.36618617	11.723759
.52002840	.44589817	.00230654	.36606627	12.592464
.51002923	.43729803	.00199508	.36596456	13.543761
.50004599	.42871658	.00171971	.36587863	14.586153
.49003355	.42011425	.00147732	.36580593	15.735750
.48009163	.41157605	.00126657	.36574522	16.993151
.47006348	.40296679	.00108094	.36569391	18.393180
.46010146	.39441679	.00092039	.36565139	19.931790
.45011067	.38584417	.00078058	.36561595	21.642277
.44018667	.37733058	.00066033	.36558674	23.529600
.43015107	.36872267	.00055540	.36556241	25.655243
.42019308	.36018251	.00046589	.36554262	28.010654
.41030184	.35170054	.00038965	.36552649	30.628093
.40030712	.34313061	.00032384	.36551325	33.595738
.39047057	.33469694	.00026870	.36550271	36.881095
.38050044	.32614924	.00022132	.36549408	40.637672
.37078717	.31782215	.00018229	.36548730	44.775833
.36073430	.30920425	.00014833	.36548174	49.637383
.35088826	.30076393	.00012053	.36547743	55.065425
.34116663	.29243045	.00009762	.36547408	61.184002
.33146455	.28411388	.00007863	.36547145	68.174751
.32133957	.27543493	.00006230	.36546932	76.585588
.31262625	.26796611	.00005070	.36546787	84.901924
.30279608	.25954004	.00003989	.36546664	95.708296
.29354415	.25160967	.00003161	.36546573	107.519620
.28450148	.24385870	.00002500	.36546504	120.905540
.27626264	.23679676	.00002005	.36546458	134.991110
.26655389	.22847491	.00001533	.36546414	154.372310
.25903206	.22202759	.00001237	.36546390	171.865060
.25130555	.21540483	.00000985	.36546369	192.533140
.24080536	.20640464	.00000715	.36546348	225.951510
.23149252	.19842220	.00000532	.36546337	261.968880
.22073661	.18920283	.00000372	.36546327	313.137030
.20190170	.17305860	.00000190	.36546317	437.510460

TABLE II. BLAST WAVE PROPERTIES FOR PLANE, CYLINDRICAL, AND SPHERICAL WAVES WITH γ = 7/5 and 5/3

d. Plane Wave (ν=1), γ = 5/3

x/x_s	u/u_s	ρ/ρ_s	p/p_s	c/c_s
1.00000000	1.00000000	1.00000000	1.00000000	1.000000
.99000004	.98505072	.95624785	.96593220	1.005051
.98000000	.97020531	.91488338	.93365708	1.010208
.97000004	.95546789	.87575597	.90307468	1.015477
.96000010	.94084226	.83872493	.87409084	1.020865
.95000006	.92633199	.80365884	.84661692	1.026378
.94000001	.91194081	.77043567	.82057006	1.032023
.93000004	.89767237	.73894217	.79587291	1.037807
.92000005	.88352987	.70907195	.77245214	1.043735
.91000002	.86951638	.68072551	.75023869	1.049817
.90000001	.85563498	.65381094	.72916842	1.056058
.89000005	.84188842	.62824164	.70918026	1.062465
.88000000	.82827901	.60393647	.69021671	1.069047
.87000002	.81480920	.58082015	.67222393	1.075811
.86000001	.80148084	.55882175	.65515085	1.082764
.85000004	.78829569	.53787520	.63894953	1.089914
.84000000	.77525502	.51791799	.62357420	1.097270
.83000001	.76236011	.49889235	.60898232	1.104838
.82000002	.74961171	.48074356	.59513321	1.112628
.80999997	.73701023	.46342028	.58198839	1.120649
.79999998	.72455605	.44687477	.56951188	1.128907
.78999993	.71224891	.43106192	.55766934	1.137414
.77999997	.70008862	.41593945	.54642829	1.146176
.76999991	.68807428	.40146760	.53575806	1.155205
.75999993	.67620513	.38760931	.52562971	1.164509
.74999990	.66447982	.37432944	.51601572	1.174098
.73999988	.65289693	.36159506	.50688992	1.183983
.73000001	.64145490	.34937559	.49822793	1.194174
.71999990	.63015135	.33764170	.49000622	1.204682
.70999996	.61898458	.32636636	.48220280	1.215520
.69999984	.60795186	.31552383	.47479666	1.226698
.68999990	.59705104	.30509038	.46776821	1.238229
.67999995	.58627923	.29504327	.46109864	1.250126
.66999999	.57563358	.28536123	.45477007	1.262404
.65999985	.56511094	.27602415	.44876557	1.275076
.64999998	.55470874	.26701384	.44306958	1.288157
.63999989	.54442325	.25831223	.43766672	1.301665
.62999985	.53425153	.24990297	.43254274	1.315615
.61999985	.52419024	.24177067	.42768425	1.330025
.60999984	.51423596	.23390066	.42307830	1.344914
.59999976	.50438522	.22627920	.41871273	1.360303

TABLE II. d. (continued)

x/x_s	u/u_s	ρ/ρ_s	p/p_s	c/c_s
.58999999	.49463495	.21889369	.41457608	1.376212
.58000001	.48498122	.21173179	.41065739	1.392665
.57000001	.47542076	.20478224	.40694625	1.409686
.56000005	.46595017	.19803451	.40343289	1.427299
.54999997	.45656591	.19147848	.40010791	1.445534
.54000000	.44726477	.18510496	.39696258	1.464420
.53000005	.43804331	.17890517	.39398845	1.483987
.52000005	.42889816	.17287075	.39117741	1.504271
.51000004	.41982607	.16699411	.38852203	1.525306
.49999999	.41082378	.16126797	.38601499	1.547134
.49000026	.40188847	.15568580	.38364957	1.569794
.48000027	.39301649	.15024095	.38141905	1.593334
.47000000	.38420481	.14492750	.37931718	1.617803
.46000021	.37545111	.13974033	.37733817	1.643253
.45000020	.36675190	.13467396	.37547631	1.669742
.44000026	.35810458	.12972373	.37372617	1.697333
.43000014	.34950623	.12488503	.37208250	1.726094
.42000019	.34095443	.12015390	.37054051	1.756098
.41000013	.33244640	.11552638	.36909537	1.787428
.40000044	.32397999	.11099911	.36774262	1.820171
.38999994	.31555182	.10656829	.36647776	1.854427
.38000004	.30716069	.10223150	.36529679	1.890299
.37000047	.29880413	.09798587	.36419571	1.927905
.36000020	.29047908	.09382843	.36317053	1.967379
.35000024	.28218428	.08975730	.36221767	2.008861
.34000022	.27391743	.08577027	.36133355	2.052512
.32999971	.26567621	.08186531	.36051473	2.098512
.31999965	.25745954	.07804113	.35975787	2.147056
.31000081	.24926627	.07429656	.35905993	2.198361
.30000144	.24109326	.07062970	.35841767	2.252687
.29000062	.23293815	.06703917	.35782806	2.310323
.28000109	.22480163	.06352510	.35728839	2.371574
.27000146	.21668109	.06008631	.35679579	2.436812
.26000204	.20857539	.05672241	.35634753	2.506452
.25000177	.20048237	.05343276	.35594105	2.580983
.24000181	.19240171	.05021761	.35557377	2.660950
.23000253	.18433254	.04707715	.35524333	2.746995
.22000282	.17627286	.04401127	.35494726	2.839879
.21000324	.16822210	.04102060	.35468327	2.940487
.20000331	.16017889	.03810565	.35444912	3.049876

TABLE II. BLAST WAVE PROPERTIES FOR PLANE, CYLINDRICAL, AND SPHERICAL WAVES WITH γ = 7/5 and 5/3

e. Cylindrical Wave (ν=2), γ = 5/3

x/x_s	u/u_s	ρ/ρ_s	p/p_s	c/c_s
1.00000000	1.00000000	1.00000000	1.00000000	1.000000
.99000002	.98260494	.92166523	.94511996	1.012644
.98000006	.96542669	.85100432	.89511844	1.025591
.97000009	.94847524	.78713225	.84950298	1.038863
.96000007	.93175984	.72927755	.80783703	1.052483
.95000002	.91528904	.67676787	.76973429	1.066474
.94000000	.89907062	.62901549	.73485168	1.080859
.93000006	.88311131	.58550493	.70288336	1.095661
.92000002	.86741644	.54578281	.67355607	1.110905
.91000004	.85199084	.50945142	.64662639	1.126614
.90000004	.83683774	.47615900	.62187548	1.142814
.89000003	.82195941	.44559531	.59910741	1.159530
.88000006	.80735703	.41748588	.57814596	1.176787
.86999999	.79303034	.39158689	.55883226	1.194611
.85999995	.77897846	.36768269	.54102353	1.213029
.84999999	.76519925	.34558121	.52459068	1.232069
.84000000	.75168948	.32511128	.50941689	1.251758
.83000007	.73844518	.30612033	.49539640	1.272126
.81999999	.72546130	.28847220	.48243349	1.293203
.80999998	.71273252	.27204546	.47044149	1.315019
.80000000	.70025258	.25673112	.45934143	1.337607
.78999995	.68801460	.24243153	.44906157	1.361000
.78000017	.67601175	.22905966	.43953704	1.385234
.77000009	.66423576	.21653616	.43070796	1.410347
.76000019	.65267930	.20479105	.42252064	1.436377
.75000005	.64133377	.19376020	.41492532	1.463365
.74000010	.63019136	.18338679	.40787716	1.491353
.73000008	.61924347	.17361896	.40133493	1.520389
.72000010	.60848179	.16441005	.39526082	1.550521
.71000016	.59789798	.15571789	.38962028	1.581799
.70000033	.58748382	.14750441	.38438176	1.614280
.69000029	.57723078	.13973474	.37951601	1.648021
.68000021	.56713100	.13237764	.37499632	1.683085
.67000017	.55717667	.12540470	.37079827	1.719538
.66000038	.54736033	.11879019	.36689934	1.757450
.65000017	.53767383	.11251007	.36327830	1.796900
.64000014	.52811052	.10654330	.35991631	1.837966
.63000019	.51866326	.10087018	.35679537	1.880737
.62000043	.50932541	.09547291	.35389918	1.925305
.61000062	.50009022	.09033503	.35121246	1.971772
.60000025	.49095098	.08544131	.34872098	2.020249

TABLE II. e. (continued)

x/x_s	u/u_s	ρ/ρ_s	p/p_s	c/c_s
.59000029	.48190255	.08077850	.34641181	2.070849
.58000017	.47293869	.07633376	.34427269	2.123698
.57000016	.46405415	.07209565	.34229238	2.178934
.56000046	.45524386	.06805356	.34046031	2.236700
.55000088	.44650267	.06419762	.33876663	2.297157
.54000036	.43782492	.06051838	.33720202	2.360485
.53000090	.42920783	.05700820	.33575818	2.426859
.52000070	.42064560	.05365870	.33442688	2.496493
.51000123	.41213543	.05046317	.33320071	2.569600
.50000133	.40367251	.04741456	.33207261	2.646431
.49000112	.39525333	.04450670	.33103586	2.727249
.48000021	.38687418	.04173370	.33008439	2.812347
.47000121	.37853403	.03909083	.32921251	2.902021
.46000109	.37022733	.03657221	.32841457	2.996646
.45000191	.36195297	.03417356	.32768542	3.096587
.44000148	.[illegible]537[illegible]48	.03188990	.32702028	3.202290
.43000259	.34545768	.02971761	.32641468	3.314193
.42000121	.3372[illegible]096	.02765183	.32586406	3.432861
.41000422	.32911977	.0[illegible]569014	.32536488	3.558787
.40000250	.32096464	.02382715	.32491281	3.692728
.39000401	.31283021	.02206090	.32450461	3.835294
.38000526	.30471187	.02038744	.32413682	3.987335
.37000562	.29660753	.01880351	.32380626	4.149761
.36000438	.28851511	.01730599	.32350992	4.323599
.35000671	.28043744	.01589270	.32324520	4.509903
.34000493	.27236703	.01455969	.32300917	4.710112
.33000831	.26431021	.01330545	.32279965	4.925515
.32001099	.25626119	.01212651	.32261421	5.157910
.31000811	.24821510	.01101974	.32245057	5.409359
.30000398	.24017854	.00998364	.32230691	5.681856
.29001115	.23214874	.00901540	.32218128	5.978021
.28000688	.22411877	.00811186	.32207181	6.301093
.27001044	.21609942	.00727188	.32197698	6.654094
.26001364	.20808353	.00649230	.32189529	7.041379
.25001894	.20007255	.00577096	.32182526	7.467682
.24002711	.19206663	.00510553	.32176562	7.938692
.23003225	.18406061	.00449333	.32171509	8.461580
.22001616	.17603958	.00393112	.32167258	9.045833
.21001715	.16803388	.00341882	.32163717	9.699401
.20007142	.16007221	.00295551	.32160810	10.431497

TABLE II. BLAST WAVE PROPERTIES FORPLANE, CYLINDRICAL, AND SPHERICAL WAVES WITH γ = 7/5 and 5/3

f. Spherical Wave (ν=3), γ = 5/3

x/x_s	u/u_s	ρ/ρ_s	p/p_s	c/c_s
1.00000000	1.00000000	1.00000000	1.00000000	1.000000
.99000010	.98017889	.88868587	.92510274	1.020283
.98000005	.96072915	.79278986	.85941096	1.041169
.97000007	.94167081	.70980266	.80161795	1.062710
.96000004	.92302074	.63767169	.75062777	1.084960
.95000008	.90479347	.57471311	.70551948	1.107972
.94000008	.88700008	.51953623	.66551378	1.131802
.93000007	.86964864	.47098843	.62994924	1.156505
.92000010	.85274409	.42811057	.59826253	1.182137
.91000003	.83628788	.39010008	.56997114	1.208755
.90000005	.82027884	.35628416	.54466125	1.236417
.89000013	.80471265	.32609564	.52197614	1.265181
.88000011	.78958210	.29905463	.50160716	1.295109
.87000009	.77487813	.27475485	.48328734	1.326264
.86000006	.76058942	.25285034	.46678460	1.358709
.85000020	.74670322	.23304612	.45189683	1.392511
.84000015	.73320466	.21508865	.43844678	1.427741
.83000012	.72007862	.19876139	.42628020	1.464473
.82000013	.70730884	.18387748	.41526138	1.502783
.81000018	.69487865	.17027558	.40527100	1.542753
.80000020	.68277089	.15781587	.39620385	1.584470
.79000024	.67096855	.14637710	.38796702	1.628024
.78000031	.65945459	.13585372	.38047832	1.673513
.77000025	.64821210	.12615344	.37366485	1.721042
.76000020	.63722480	.11719567	.36746182	1.770721
.75000028	.62647690	.10890978	.36181165	1.822669
.74000025	.61595267	.10123320	.35666263	1.877013
.73000041	.60563763	.09411123	.35196893	1.933888
.72000028	.59551704	.08749490	.34768897	1.993444
.71000058	.58557806	.08134153	.34378601	2.055833
.70000028	.57580676	.07561210	.34022620	2.121232
.69000066	.56619216	.07027315	.33698003	2.189814
.68000064	.55672163	.06529344	.33401982	2.261785
.67000068	.54738468	.06064579	.33132122	2.337352
.66000074	.53817101	.05630546	.32886178	2.416748
.65000052	.52907066	.05225007	.32662122	2.500322
.64000082	.52007530	.04845981	.32458124	2.588039
.63000073	.51117564	.04491603	.32272474	2.680497
.62000100	.50236441	.04160239	.32103651	2.777909
.61000092	.49363361	.03850352	.31950241	2.880625
.60000139	.48497713	.03560581	.31810961	2.989014

TABLE II. f. (continued)

x/x_s	u/u_s	ρ/ρ_s	p/p_s	c/c_s
.59000108	.47638741	.03289616	.31684624	3.103499
.58000109	.46785946	.03036319	.31570148	3.224516
.57000139	.45938773	.02799607	.31466540	3.352555
.56000187	.45096704	.02578479	.31372883	3.488151
.55000207	.44259225	.02371996	.31288321	3.631901
.54000255	.43425951	.02179310	.31212092	3.784442
.53000300	.42596453	.01999610	.31143470	3.946488
.52000296	.41770323	.01832136	.31081793	4.118832
.51000360	.40947321	.01676204	.31026461	4.302318
.50000413	.40127068	.01531139	.30976904	4.497919
.49000558	.39309365	.01396332	.30932608	4.706671
.48000500	.38493709	.01271156	.30893084	4.929820
.47000635	.37680187	.01155107	.30857904	5.168589
.46000640	.36868322	.01047620	.30826653	5.424518
.45000758	.36058111	.00948223	.30798963	5.699184
.44001103	.35249470	.00856450	.30774498	5.994380
.43000790	.34441521	.00771775	.30752914	6.312446
.42000973	.33635053	.00693868	.30733945	6.655345
.41001137	.32829518	.00622277	.30717321	7.025864
.40001846	.32025252	.00556648	.30702800	7.426746
.39001779	.31221086	.00496526	.30690141	7.861914
.38002020	.30417799	.00441626	.30679161	8.334781
.37002481	.29615235	.00391601	.30669660	8.849772
.36003052	.28813232	.00346123	.30661478	9.411986
.35002743	.28010929	.00304842	.30654449	10.027885
.34002148	.27208745	.00267496	.30648444	10.703968
.33003543	.26408454	.00233883	.30643347	11.446393
.32006348	.25609390	.00203691	.30639036	12.264529
.31004889	.24807429	.00176527	.30635393	13.173630
.30004625	.24006451	.00152294	.30632348	14.182324
.29006541	.23207367	.00130779	.30629813	15.303927
.28008761	.22408649	.00111712	.30627717	16.557921
.27010540	.21609682	.00094876	.30625995	17.966596
.26018978	.20816126	.00080176	.30624599	19.543910
.25021880	.20018208	.00067246	.30623459	21.339906
.24024027	.19219740	.00055991	.30622543	23.386206
.23025386	.18421167	.00046258	.30621815	25.728897
.22038949	.17631430	.00037980	.30621245	28.394373
.21058596	.16847070	.00030947	.30620806	31.455399
.20062742	.16050328	.00024885	.30620456	35.077849